Analytic solutions to two quaternion attitude estimation problems

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Abstract—**This paper presents solutions to the following two common quaternion attitude estimation problems: (i) estimation of attitude using measurement of two reference vectors, and (ii) estimation of attitude using rate measurement and measurement of a single reference vector. Both these problems yield to a direct geometric analysis and solution. The former problem already has a well established analytic solution in literature using linear algebraic methods. This note shows how the solution may also be obtained using geometric methods, which are not only more intuitive, but also amenable to unconventional extensions. With respect to the latter problem, existing solutions typically involve filters and observers and use a mix of differential-geometric and control systems methods. However, no analytic solution has yet been reported to this problem. In this note, both the problems are formulated as optimization problems, which can be solved analytically to obtain a unique closed-form solution. The analytic attitude estimates are (i) instantaneous with respect to the measurements, thus overcoming the latency inherent in solutions based upon negative feedback upon an error, which can at best show asymptotic convergence, (ii) exact, thus overcoming errors in solutions based upon linear methods, and (iii) geometry-based, thus enabling imposition of geometric inequality constraints.**

I. Introduction

The problem of estimating the attitude of a rigid body with respect to a reference coordinate system, by measuring reference vectors in a body-fixed frame, originally posed by G. Wahba in [1], has been treated abundantly in literature. Multiple solutions have been reported for Wahba's problem: using polar decomposition [2], Davenport's q-method [3], an SVD method, a three-axis attitude estimator TRIAD [4], the Quaternion estimator QUEST [5], *etc*.

Although both Davenport's q-method and QUEST use the quaternion representation of attitude, they ultimately reduce to an eigenvalue-eigenvector problem. Thus it can be seen that all solutions are linear algebraic in nature, and given the vast array of tools available for linear problems, they are all readily solved. This advantage is, however, associated with the accompanying weakness that it is not straightforward to incorporate nonlinear and nonholonomic constraints in the problem. For instance, in [6], the authors describe the attitude control of a spaceshuttle during a docking operation, when there is a hard constraint with respect to a nominal pitch angle in order to ensure that a trajectory control sensor is oriented towards the target platform. The attitude guidance module then estimates an optimal pitch attitude that complies with the hard constraint and minimizes the control effort. Similarly, in [7], the authors describe a reference governor with a pointing inclusion constraint such that the spacecraft points towards a fixed target, or an exclusion constraint such that sensitive equipment is not exposed to direct solar radiation. Such inequality constraints are obviously nonholonomic, and while being quite common in practice, are notoriously difficult to incorporate in a linear algebraic solution. Once the guidance or reference module determines an attitude that complies with the constraints, a controller module is used to achieve bounded or asymptotic stability with respect to the reference.

Relatedly, the advent of small unmanned vehicles has motivated the development of solutions that depend upon minimal measurement resources in order to reduce the weight and cost of the sensor payload. In particular, it is of considerable interest to estimate the attitude using a single vector measurement, possibly supplemented by a rate measurement, thus leading us to the second of the stated problems. This interest is partly fueled by the availability of cheap commercial-off-the-shelf inertial measurement units (IMUs) that contain MEMS-based gyroscopes and accelerometers [8]. The research is also partly fueled by the realization that attitude estimation and control is a key challenge in the design of small autonomous aerial robots [9].

The second problem is most frequently solved using an extended Kalman filter (EKF) [10]. The EKF provides a point-wise attitude estimate and is instantaneous with respect to the measurements. However, resulting from linearization of an intrinsically nonlinear problem, this solution is not robust to large changes in the attitude state [11].

More recently, some solutions have been reported in literature which use nonlinear observers or filters to solve the single-vector measurement problem [11], [12], [13], [14], [15]. These solutions have typically used an appropriate error signal in negative feedback to estimate the attitude. The solutions in [11], and [13] are quite general, and while having been developed for multiple vector measurements, they extend smoothly to the case of a single vector measurement. The solutions presented in [14], and [15] are more specific to the availability of single vector measurements. A common characteristic in this group of solutions is the use of negative feedback from an error signal to estimate the attitude and an (a-priori) unknown gain, that needs to be tuned in order to achieve satisfactory estimator performance. Such an estimator is bound to have a finite latency with respect to the input, and cannot instantaneously track abrupt or discontinuous changes in the measurements, and the convergence of the estimate to the true attitude is at best asymptotic.

In contrast to the linear algebraic and filter approaches available in literature, this paper analyzes the attitude estimation

The authors would like to gratefully acknowledge partial support from the Air Force Office of Scientific Research, and the Office of Naval Research.

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problems from a geometric perspective. In the process, we obtain solutions that overcome some of the shortcomings in the previous solutions. Firstly, being of a geometric nature, the solutions easily extend to problems involving geometric constraints, irrespective of whether they are holonomic equations or nonholonomic inequality constraints. Secondly, the analytic solutions provide an instantaneous estimate for the attitude which is consistent with respect to the vector measurement at every time step. Besides the mathematical elegance of having an analytic solution, this also has several applications in autonomous guidance, navigation, and control systems: it enables the deployment of frugal single-vector-measurement sensor-suites, and the zero-latency accuracy of the solution is useful in multiple-vector-measurement suites in overcoming sudden failures or intermittent losses in some of the components without leading to large transient errors that could potentially cause system breakdown.

A brief outline of the paper is as follows. We begin by introducing the geometric approach and formulating the stated problems in the language of mathematics in section II. The next section, section III-A, presents the solution to the first problem, and relates it to the existing solutions from literature. The next section, Section III-B, solves the second problem and also provides results relating to the accuracy of the solution, and an estimation of the bias errors in the rate measurement. This is followed by verification of the theory using simulations in section IV.

II. Notation, definitions, and problem formulation

In this section, we describe the geometry associated with vector measurements and formulate the attitude estimation problems as well-posed mathematical problems.

The attitude of the rigid body with respect to a reference coordinate system shall be represented in the form of the quaternion $\check{q} = [q_0 \;\; q_1 \;\; q_2 \;\; q_3]^T$ which is constrained to have unit magnitude: $\check{q}^T\check{q} = 1$, or $\check{q} \in \mathbb{S}^3$, the unit 3-sphere. The quaternion components are related to the axis-angle representation of a rotation by the relation $q_0 = \cos\Phi/2$, and $[q_1 \;\; q_2 \;\; q_3]^T = n\sin\Phi/2$, for a rotation through Φ about the axis n. The product of two quaternions $\check{q}$ and $\check{p}$ shall be denoted as $\check{q} \otimes \check{p}$. For a discussion on the quaternion based attitude representation, the reader is referred to [16] chapter 11.

A reference vector, $\mathbf{h}$, shall be defined as a unit magnitude vector that points in a specified direction. Examples include the direction of fixed stars relative to the body, the Earth's magnetic field, gravitational field *etc*. The components of any such vector lie on a 2-sphere $\mathbb{S}^2$, and may be measured in any three-dimensional orthogonal coordinate system. In the context of our problems, two obvious choices for the coordinate system are the reference coordinate system (relative to which the rigid body's attitude is to be determined), and a coordinate system fixed in the body. We assume the availability of measurement apparatus to obtain the vector's components in a three-dimensional orthogonal coordinate system, $h \in \mathbb{S}^2 \subset \mathbb{R}^3$ in the reference coordinate system, and $b \in \mathbb{S}^2 \subset \mathbb{R}^3$ in the body-fixed coordinate system.

Suppose we have a vector measurement available at our disposal. What information does the measurement of b (and knowledge of h) provide regarding the body's attitude relative to the reference coordinate system? A rotation quaternion (or, for that matter, any rotation representation) has three scalar degrees of freedom. We see that we have 3 scalar measurements in b that are related to h in terms of the rotation quaternion. However, we also know that the measurement would retain the magnitude of $\mathbf{h}$, *i.e.*, $\|\mathbf{h}\|^2 = h^T h = b^T b = 1$, so there is one scalar degree of redundancy in our measurement and only two scalar degrees of information. Reconciling with this redundancy, we can therefore isolate the quaternion from a three-dimensional set of possibilities to a single-dimensional set.

The redundancy can be visualized as shown in figure 1. The measurement of a single vector in body-fixed axes confines the body's attitude to form a conical solid of revolution about $\mathbf{h}$: those and only those attitudes on the cone would yield the same components b. We shall refer to the set of attitude quaternions consistent with a measurement as the "feasibility cone" Q_b corresponding to that measurement b, *i.e.*, the measurement confines the attitude quaternion $\check{q}$ to lie in Q_b. From the previous discussion, Q_b is one-dimensional and $\check{q}$ has effectively a single degree of freedom. We shall repeatedly draw intuition from the geometry in figure 1 to guide us in the solutions to the stated problems.

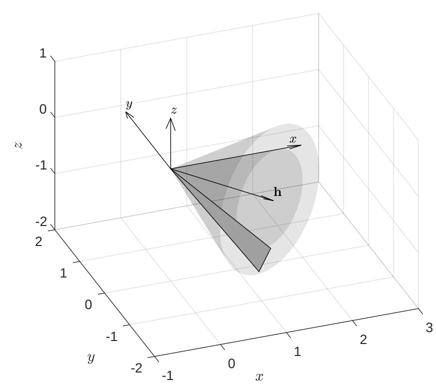

Fig. 1. Possible attitudes of a minimal rigid body formed out of three non collinear points (represented by the triangular patch) consistent with a measurement of a single vector $\mathbf{h}$.

A. *Problem 1. Estimation using measurements of two reference vectors*

Let the components of two vectors $\mathbf{h}$ and $\mathbf{k}$ be $a = [a_1 \;\; a_2 \;\; a_3]^T$ and $b = [b_1 \;\; b_2 \;\; b_3]^T$ in the body coordinate system, and $h = [h_1 \;\; h_2 \;\; h_3]^T$ and $k = [k_1 \;\; k_2 \;\; k_3]^T$ in the reference coordinate system respectively. As described above, each reference vector measurement provides two scalar degrees of information regarding the attitude of the rigid body. It is immediately clear that the problem is overconstrained, and we have more equations than unknowns. Geometrically, we have two feasibility cones Q_a and P_b, with the body-axes intersecting along two lines, but with different roll angles for the body about the body-axis. Thus there is no exact solution to this problem in general, unless some of the measurement information is redundant or discarded.

A trivial means to well-pose the problem is to discard components of one of the vector, say $\mathbf{k}$, along the second, $\mathbf{h}$. This is exactly what is done with the TRIAD solution [4], where we use the orthogonal vector triad $\mathbf{h}$, $\mathbf{h} \times \mathbf{k}$, and $\mathbf{h} \times (\mathbf{h} \times \mathbf{k})$ to determine the attitude. A more sophisticated approach is to use all the measurement information – four scalar degrees of information with two reference vector measurements – , and frame the problem as a constrained four-dimensional optimization problem in terms of the quaternion components. This leads to Davenport's q-method and QUEST solutions to Wahba's problem [1].

A novel third approach presented in this paper, is to first determine two solutions $\check{q}$ and $\check{p}$, one each lying on each of the feasibility cones Q_a and P_b corresponding to the measurements a and b, and "closest" to the other cone in some sense. We then fuse the estimates $\check{q}$ and $\check{p}$ appropriately to obtain the final attitude estimate. For example, the final estimate could be obtained using linear spherical interpolation, and the weights be chosen to represent the relative significance attached to the individual measurements.

The first problem can therefore be stated as: *given the measurements a and b of the two reference vectors h and k in a rotated coordinate system, we would like to estimate the rotated system's two attitude quaternions $\check{q} \in Q_a$ closest (in the least squares sense) to P_b and $\check{p} \in P_b$ closest (in the least squares sense) to Q_a, where Q_a and P_b are the respective feasibility cones.*

B. Problem 2. Estimation using rate measurement and measurement of single vector

Suppose we have a measurement of the components $\omega = [\omega_1\ \omega_2\ \omega_3]^T$ of the angular velocity $\boldsymbol{\omega}$ of a moving rigid body, and that we also have a measurement of the components $b = [b_1\ b_2\ b_3]^T$ of a reference vector $\mathbf{h}$, both measurements being made in the body coordinate system. The components of $\mathbf{h}$ in the reference coordinate system are also known, say $h = [h_1\ h_2\ h_3]^T$. The problem is to make a "best" estimate of the body's attitude $\check{q}$ on the basis of the pair of measurements ω and b, and knowing h.

Without loss of generality, the initial attitude quaternion may be assumed to be $\check{1}$. The angular velocity ω can be forward integrated to obtain a "dead-reckoning" estimate of the rotation quaternion. We start with the initial attitude, $\check{1}$, and then integrate the differential kinematic equation, to obtain the integrated estimate $\check{p}$. On account of errors in the measurement of ω, this differs from the actual attitude $\check{q}$ of the body. Since we are integrating the errors, the attitude estimates are expected to diverge with time and lead to what is referred to as "drift" in the predicted attitude estimate. Constant errors in the measurement lead to a drift that is proportional to the time of integration, while random white wide-sense stationary noise leads to a drift that is proportional to the square-root of time [17]. Let the error in ω be denoted by the unknown signal $e(t) \in \mathbb{R}^3$ in the body coordinate system. The integrated estimate also has three scalar degrees of error, though it may depend upon e in some complicated path-dependent form.

The second measurement available is b – and of course the knowledge of its reference axes components h. As described at the beginning of this section, this provides two additional scalar degrees of information besides the three from the rate measurement, and constrains the attitude $\check{q}$ to lie in the feasibility cone Q_b. In order to determine the six scalar unknowns, three related to the attitude $\check{q}$, and three related to the integration of the rate measurement error e, we are still lacking one scalar degree of information. In order to specify this degree of freedom and close the problem, we now impose a sixth scalar constraint that uses the attitude $\check{p}$ that was obtained by integrating the kinematic differential equation. We choose that particular $\check{q} \in Q_b$ which is best in the sense that it deviates the least from $\check{p}$.

To summarize, the second problem is *to estimate the attitude quaternion $\check{q}$ which would yield the measurement b in the rotated coordinate system for the reference vector h, and closest (in the least squares sense) to the estimate $\check{p}$ obtained by integrating the angular velocity measurement ω as given in the kinematic differential equation.*

C. Nature of measurements of reference vector and angular velocity

The reference vector measurements are assumed to have random, unbiased noise in each of the components, but that they are subsequently normalized for unit magnitude before being passed on to the attitude estimator. This is the most common situation in practice. Any deterministic errors in the measurement are also assumed to be compensated for, *e.g.* acceleration compensation in gravity sense, local field compensation in magnetic field sense.

The angular velocity measurement is assumed to have random, unbiased noise in each of the components, and additionally have an independent, constant (or of negligible variation with time) bias error [13]. Deterministic errors in this measurement are also assumed to be compensated for. The angular velocity is not of unit magnitude, in general.

Having laid the groundwork for both the problems, the detailed solutions follow in the next section.

III. Attitude quaternion estimation

We first motivate the use of quaternions for attitude representation by establishing the equivalence between attitude deviations and angles in the following lemma.

Lemma 1. *The Euclidean distance $\|\check{q} - \check{1}\|$ of an attitude quaternion, $\check{q} = [c_{\Phi/2}\ s_{\Phi/2} n]^T$, from the identity element, $\check{1}$, is a positive definite and monotonic function of the magnitude of the principal angle of rotation Φ.*

Proof.

$$\|\check{q} - \check{1}\|^2 = (c_{\Phi/2} - 1)^2 + s^2_{\Phi/2} = 2(1 - c_{\Phi/2}) = 4\sin^2(\Phi/4),$$

which is a positive definite monotonic function of Φ in the interval $[-2\pi, 2\pi]$. □

We next provide two particular solutions for the simpler problem of estimating the attitude quaternion using a single reference vector measurement, in lemma 2. We note the algebraic constraint imposed by a vector measurement on the attitude quaternion $\check{q}$. The quaternion $\check{q}$ represents a rigid body rotation, and it transforms the components of the reference vector from h in the reference coordinate system to b in the body-fixed coordinate system:

$$\check{h} = \check{q} \otimes \check{b} \otimes \check{q}^{-1}$$

or

$$\check{q} \otimes \check{b} = \check{h} \otimes \check{q}\,, \tag{1}$$

where the checked quantities $\check{h} = [0\ h^T]^T$ and $\check{b} = [0\ b^T]^T$ are the quaternions corresponding to the 3-vectors h and b. Equation (1) expresses the vector measurement constraint as a linear equation in $\check{q}$.

Lemma 2. *Suppose the components of a reference vector are given by h and b in the reference and body coordinate systems respectively. Let $\Phi = \operatorname{acos} b^T h$, $c = \cos\Phi/2 = \sqrt{(1+b^T h)/2}$ and $s = \sin\Phi/2 = \sqrt{(1-b^T h)/2}$. Then, two particular solutions for the body's attitude are given by:*

$$\check{r} = \begin{bmatrix} c \\ s(b\times h)/\|b\times h\| \end{bmatrix} \ \textit{and}\ \check{s} = \begin{bmatrix} 0 \\ (b+h)/\|b+h\| \end{bmatrix}. \tag{2}$$

Proof. These two solutions are orthogonal in quaternion space, and correspond to the smallest and largest single axis rotations in $[0,\pi]$ that are consistent with the vector measurement in three-dimensional Euclidean space. Geometrically, the first is a rotation through $\operatorname{acos}(b^T h)$ about $(b\times h)/\|b\times h\|$, the second is a rotation through π about $(b+h)/\|b+h\|$. Noting that $\|b\times h\| = \|b\|\|h\|\sin\Phi = \|b\|\|h\|2sc$, and $\|b+h\| = 2c$, we obtain

$$\begin{aligned}
\begin{bmatrix} c \\ (b\times h)/(2c) \end{bmatrix} \otimes \begin{bmatrix} 0 \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ cb + (h - bb^T h)/(2c) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ (b+h)/2c \end{bmatrix} = \begin{bmatrix} 0 \\ h \end{bmatrix} \otimes \begin{bmatrix} c \\ (b\times h)/(2c) \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{bmatrix} 0 \\ (b+h)/(2c) \end{bmatrix} \otimes \begin{bmatrix} 0 \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ (h\times b)/(2c) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ h \end{bmatrix} \otimes \begin{bmatrix} 0 \\ (b+h)/(2c) \end{bmatrix},
\end{aligned}$$

which completes the proof. As a clarification, when $b \to h$, $\check{r}$ and $\check{s}$ are assumed to take the obvious limits, $\check{1}$ and $\check{h}$, and when $b \to -h$, they are assumed to take the obvious limits, $\check{i} = [0\ i]^T$ and $\check{j} = [0\ j]^T$, where $[h\ i\ j]$ is an orthogonal vector triplet. □

The two special solutions can be rotated by any arbitrary angle about the reference vector h and we would still lie within the feasibility cone, as shown in the next lemma.

Lemma 3. *If $\check{q}$ lies in the feasibility cone Q_b of the measurement b for the reference vector h, then so does any attitude quaternion obtained by rotating $\check{q}$ through an arbitrary angle about h. Conversely, all attitude quaternions lying on the feasibility cone are related to each other by rotations about h.*

Proof. Let Φ be any angle, and let $\check{p}$ be $\check{q}$ rotated through Φ about h, *i.e.*,

$$\check{p} = \begin{bmatrix} c \\ sh \end{bmatrix} \otimes \check{q}\,,$$

where $c = \cos\Phi/2$ and $s = \sin\Phi/2$. Then,

$$\begin{aligned}
\check{p} \otimes \check{b} &= \begin{bmatrix} c \\ sh \end{bmatrix} \otimes \check{q} \otimes \check{b} = \begin{bmatrix} c \\ sh \end{bmatrix} \otimes \check{h} \otimes \check{q} \\
&= \check{h} \otimes \begin{bmatrix} c \\ sh \end{bmatrix} \otimes \check{q} = \check{h} \otimes \check{p}\,.
\end{aligned}$$

where we have used the fact that two nonzero rotations commute if and only if they are about the same axis. Conversely,

$$\begin{aligned}
\check{q}^{-1} \otimes \check{h} \otimes \check{q} &= b = \check{p}^{-1} \otimes \check{h} \otimes \check{p} \\
\Rightarrow \check{p} \otimes \check{q}^{-1} \otimes \check{h} &= \check{h} \otimes \check{p} \otimes \check{q}^{-1} \\
\Rightarrow \check{p} \otimes \check{q}^{-1} &= \begin{bmatrix} c \\ sh \end{bmatrix},
\end{aligned}$$

which completes the proof. □

Thus, we already see that we have a one dimensional infinity of possible solutions for the attitude quaternion if we have a single reference vector measurement. In fact, the two special solutions provided in lemma 2 are rotations of each other about h through π. In order to obtain a unique solution, we could add either another vector measurement (Wahba's problem), or include an angular velocity measurement.

A. Attitude estimation using two vector measurements

We now derive a unique solution for the attitude quaternion when we have measurements of two reference vectors and would like to incorporate both of them in deriving the attitude estimate. Let a and b be the body-referred components of reference vectors h and k ($h, k \in \mathbb{S}^2$ contain the components of the two vectors along some reference coordinate axes) respectively. Suppose the rotation quaternion is estimated to be $\check{q} = [q_0\ q]^T$ on the basis of a, and it is independently estimated to be $\check{p} = [p_0\ p]^T$ on the basis of b.

The estimates $\check{q}$ and $\check{p}$ are each indeterminate to one scalar degree of freedom as shown in lemma 3: a rotation about the corresponding vectors h and k respectively. Let these rotations be given by the quaternions $\check{r}_1 = [c_1\ s_1 h]^T$ and $\check{r}_2 = [c_2\ s_2 k]^T$ respectively where $c_i = \cos\Phi_i/2$ and $s_i = \sin\Phi_i/2$ for $i \in \{1,2\}$. The problem is to determine the optimal values of Φ_1 and Φ_2 so as to minimize the displacement from the rotated $\check{r}_1 \otimes \check{q}$ to $\check{r}_2 \otimes \check{p}$.

$$\begin{aligned}
\check{r}_1 \otimes \check{q} &= \begin{bmatrix} c_1 \\ s_1 h \end{bmatrix} \otimes \begin{bmatrix} q_0 \\ q \end{bmatrix} = \begin{bmatrix} c_1 q_0 - s_1 q^T h \\ c_1 q + s_1 q_0 h + s_1 h \times q \end{bmatrix}, \\
\check{r}_2 \otimes \check{p} &= \begin{bmatrix} c_2 \\ s_2 k \end{bmatrix} \otimes \begin{bmatrix} p_0 \\ p \end{bmatrix} = \begin{bmatrix} c_2 p_0 - s_2 p^T k \\ c_2 p + s_2 p_0 k + s_2 k \times p \end{bmatrix}.
\end{aligned} \tag{3}$$

We could either minimize $\|\check{r}_1 \otimes \check{q} - \check{r}_2 \otimes \check{p}\|^2$, or equivalently, maximize the first component of $(\check{r}_1 \otimes \check{q})^{-1} \otimes \check{r}_2 \otimes \check{p}$. In order to keep the reasoning straightforward, we choose the former. So we need to minimize the cost function

$$J(\Phi_1, \Phi_2) = (c_1 q_0 - s_1 q^T h - c_2 p_0 + s_2 p^T k)^2$$

$$
\begin{aligned}
&\quad + \|c_1 q + s_1(q_0 h + h \times q) - c_2 p - s_2(p_0 k + k \times p)\|^2 \,, \\
&= c_1^2(q_0^2 + q^T q) + s_1^2((q^T h)^2 + \|q_0 h - q \times h\|^2) \\
&\quad + c_2^2(p_0^2 + p^T p) + s_2^2((p^T k)^2 + \|p_0 k - p \times k\|^2) \\
&\quad - 2c_1 c_2 (q_0 p_0 + q^T p) \\
&\quad - 2s_1 s_2 (q^T h p^T k + (q_0 h - q \times h)^T (p_0 k - p \times k)) \\
&\quad + 2c_1 s_1 (-q_0 q^T h + q_0 q^T h - q^T q \times h) \\
&\quad + 2c_2 s_2 (-p_0 p^T k - p_0 p^T k + p^T p \times k) \\
&\quad + 2c_1 s_2 (q_0 p^T k - p_0 q^T k + q^T p \times k) \\
&\quad + 2c_2 s_1 (p_0 q^T h - q_0 p^T h + p^T q \times h) \\
&= 2 + 2ac_1c_2 + 2bs_1s_2 + 2cc_1s_2 + 2ds_1c_2 \,, \qquad (4)
\end{aligned}
$$

where $a = -q_0 p_0 - q^T p$, $b = (-q_0 p^T + p_0 q^T - (q \times p)^T) h \times k - (q_0 p_0 + q^T p) h^T k$, $c = k^T(q_0 p - p_0 q + q \times p)$, and $d = h^T(p_0 q - q_0 p + p \times q)$, are known quantities. Now minimizing the cost function with respect to the independent pair of variables $\Phi_1 + \Phi_2$ and $\Phi_1 - \Phi_2$ yields

$$
\begin{bmatrix} \Phi_1 - \Phi_2 \\ \Phi_1 + \Phi_2 \end{bmatrix} = 2 \begin{bmatrix} \operatorname{atan2}(c - d, -(a + b)) \\ \operatorname{atan2}(-(c + d), b - a) \end{bmatrix} . \qquad (5)
$$

Equation (5) can be solved for Φ_1, and Φ_2, and that completes the solution. The above derivation can be summarized in the form of the following theorem:

Theorem 4. *If $\check{q}$ and $\check{p}$ are any two special attitude estimates for a rotated system, derived independently using the measurements a and b in the body-fixed coordinate system of two linearly independent reference vectors h and k respectively, then the optimal estimate incorporating the measurement b in $\check{q}$ is $\check{r}_1 \otimes \check{q}$, and the optimal estimate incorporating the measurement a in $\check{p}$ is given by $\check{r}_2 \otimes \check{p}$, where $\check{r}_1 = [c_1 \;\, s_1 h]^T$ and $\check{r}_2 = [c_2 \;\, s_2 k]^T$, and c_1, c_2, s_1, and s_2 are given by equation (5).*

Proof. The proof follows from the construction leading to equations (3, 5). Refer figure 2. ☐

Remark 4.1. *Sign indeterminacy*: The solution for c_1, s_1, c_2, and s_2 in equation (5) involves taking a square-root, but once the sign of the square-root is chosen for one of the four quantities, it gets decided for the other three. Both of the resulting attitude quaternions represent the same rotation in three-dimensional Euclidean space.

Remark 4.2. *Relation to the TRIAD attitude estimate*: The attitude estimates $\check{r}_1 \otimes \check{q}$ and $\check{r}_2 \otimes \check{p}$, where $\check{r}_1 = [c_1 \;\, s_1 h]^T$ and $\check{r}_2 = [c_2 \;\, s_2 k]^T$, are the same as the TRIAD solution in literature [18]. Each of them individually yields an estimate that is competely consistent with one measurement, but only partially consistent with the other.

Remark 4.3. *Relation to the solutions of Wahba's problem*: In order to obtain the solution to Wahba's problem [3], [18], we could now interpolate between the two solutions obtained in equations (3, 5). Let $\check{q}, \check{p}$ be unit quaternions and $x \in \mathbb{R} \in [0, 1]$. The interpolated quaternion from $\check{q}$ to $\check{p}$ is given by any of the following four equivalent expressions [19]:

$$
\begin{aligned}
&\check{q} \otimes (\check{q}^{-1} \otimes \check{p})^x = \check{p} \otimes (\check{p}^{-1} \otimes \check{q})^{1-x} \\
&= (\check{q} \otimes \check{p}^{-1})^{1-x} \otimes \check{p} = (\check{p} \otimes \check{q}^{-1})^x \otimes \check{q} \,. \qquad (6)
\end{aligned}
$$

The scalar x is now choosen to perform a desired weighting of the two estimates $\check{q}$ and $\check{p}$ in the final result. When the noise in each of the measurements a and b is random and unbiased with variance σ_i^2, the appropriate choice for x would be $\sigma_a^2/(\sigma_a^2 + \sigma_b^2)$. The resulting estimate is then the same as that obtained using Davenport's q-method.

Remark 4.4. *Incorporating hard inequality constraints*: Since the presented solution is geometric in nature, it is straightforward to include geometric constraints on the solution. For instance, some attitude estimation problems have hard constraints [6], [7]. In control solutions, such constraints are most often enforced using Barrier Lyapunov functions (BLFs) [20] for bounded solutions. Such a strategy can easily be employed in our framework, in contrast with the linear algebraic solutions which are more suitable to handle quadratic forms. Instead of determining the interpolaton factor x using the noise variance, it can be determined as the argument that minimizes a cost function that contains a BLF:

$$
x = \underset{x \in [0,1]}{\operatorname{argmin}} (\alpha \sec(x/a) + (1 - x)^2), \qquad (7)
$$

where α and a are appropriately chosen constants. It may be appreciated that the cost function can be any infinite potential well, and not just the above formulation. This generality is enabled by the simple interpolation of the geometric angle between the two solutions of theorem 4.

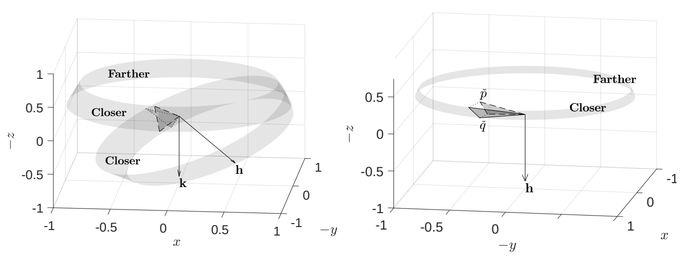

Fig. 2. A visual depiction of the solutions presented in theorems 3 and 4. The image on the left shows the two solutions $\check{r}_1 \otimes \check{q}$ (dotted triangle) and $\check{r}_2 \otimes \check{p}$ (dashed triangle) of theorem 3. The figure on the right shows the solution $\check{q}$ (solid triangle) of theorem 4 obtained by projecting the integrated attitude $\check{p}$ (dashed triangle) onto the feasibility cone of vector measurement b.

B. Attitude estimation using single vector measurement and rate measurement

We first write down the constraints imposed by the measurement upon the attitude quaternion $\check{q} = [c \;\, s[\mathbf{n}]]^T = [c \;\, sn_1 \;\, sn_2 \;\, sn_2]^T$, where $c = \cos(\Phi/2)$ and $s = \sin(\Phi/2)$ are functions of the rotation angle Φ, and $\mathbf{n}$ is a unit vector along the rotation axis with components $n = [n_1 \;\, n_2 \;\, n_3]^T$ in the reference coordinate system. The constraint is given in equation (1). Converting the quaternion multiplication to vector notation, equation (1) can also be written as:

$$
\begin{bmatrix} -sn^T b \\ cb + s[n\times]b \end{bmatrix} = \begin{bmatrix} -sh^T n \\ ch + s[h\times]n \end{bmatrix} ,
$$

i.e.,

$$
\begin{bmatrix} -s(h - b)^T n \\ c(h - b) + s[(h + b)\times]n \end{bmatrix} = 0 \,,
$$

where $[n\times]$ denotes the cross product matrix associated with the 3-vector n. Expanding the vectors,

$$\begin{bmatrix} & -f_1 & -f_2 & -f_3 \\ f_1 & & -g_3 & g_2 \\ f_2 & g_3 & & -g_1 \\ f_3 & -g_2 & g_1 & \end{bmatrix} \begin{bmatrix} c \\ sn_1 \\ sn_2 \\ sn_3 \end{bmatrix} = 0 \ , \tag{8}$$

where $\qquad f = h - b \qquad$ and $\qquad g = h + b$,

so that $\qquad f_1 g_1 + f_2 g_2 + f_3 g_3 = f^T g = h^T h - b^T b = 0$.

While it is not obvious, equation (8) has a double redundancy, so the system of four linear equations actually has rank 2 and nullity 2. This can be seen by considering the solution:

$$\check{q} = \begin{bmatrix} c \\ (-cf_2 + sn_3 g_1)/g_3 \\ (cf_1 + sn_3 g_2)/g_3 \\ sn_3 \end{bmatrix} \ , \tag{9}$$

where sn_1 and sn_2 are solved in terms of c and sn_3 using the inner two row equations in equation (8). Substituting these in the outer two rows of equation (8) satisfies them trivially, so these two rows do not yield any additional information. This makes sense as we have not yet imposed the normalization constraint that $n_1^2 + n_2^2 + n_3^2 = 1$ (c and s, representing $\cos\Phi/2$ and $\sin\Phi/2$, are already assumed to satisfy $c^2 + s^2 = 1$). And we are anyway to end up with one degree of freedom in $\check{q}$ if using the vector measurement constraint alone, as discussed earlier.

We could apply the normalization constraint,

$$n_1^2 + n_2^2 + n_3^2 = 1 \ , \tag{10}$$

at this point to express n completely in terms of Φ:

$$\frac{c^2 f_2^2}{s^2 g_3^2} + \frac{g_1^2}{g_3^2} n_3^2 + \frac{c^2 f_1^2}{s^2 g_3^2} + \frac{g_2^2}{g_3^2} n_3^2 + 2\frac{c}{s g_3^2} n_3 (f_1 g_2 - f_2 g_1) \\ + n_3^2 = 1 \ ,$$

or,

$$n_3^2 g^T g + 2\frac{c}{s} n_3 (f_1 g_2 - f_2 g_1) + \frac{c^2}{s^2}(f_1^2 + f_2^2) = g_3^2. \tag{11}$$

The above quadratic equation can be solved for n_3 in terms of $c/s = \cot\Phi/2$ to yield:

$$n_3 = -\frac{c(f_1 g_2 - f_2 g_1)}{s g^T g} \\ \pm \sqrt{\frac{c^2((f_1 g_2 - f_2 g_1)^2 - g^T g (f_1^2 + f_2^2))}{(s g^T g)^2} + \frac{g_3^2}{g^T g}} \ . \tag{12}$$

The above equation in conjunction with the inner two rows of equation (9) expresses all three components of n in terms of $c/s = \cot(\Phi/2)$ and the measured quantities f and g. Thus we are left with the single degree of freedom, Φ, in $\check{q}$, as expected. However, as shall be seen later, it is easier to retain n_3 as a variable in our problem, along with the normalization constraint 11.

We now move on to utilizing the angular velocity measurement that determines the differential evolution of the attitude. The kinematic differential equation for the quaternion is:

$$\dot{\check{q}} = \frac{1}{2}\check{q} \otimes \check{\omega} = \frac{1}{2} \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \ , \tag{13}$$

where $\otimes$ indicates quaternion multiplication, and $\check{\omega}$ is the quaternion form of the 3-vector ω. In discrete time, denoting the integrated estimate as $\check{p}_{i+1}$, the above equation takes the form

$$\check{p}_{i+1} = \check{q}_i + \frac{T}{2}\check{q}_i \otimes \check{\omega}_i \ , \tag{14}$$

where T is the time step from the previous estimation of $\check{q}_i$ to the current estimation $\check{p}_{i+1}$. In the subsequent derivation, we shall omit the subscript of $\check{p}$, as there is no ambiguity.

The deviation of the vector-aligned quaternion estimate, $\check{q}$ in equation (9), from the integrated estimate, $\check{p}$ in equation (14), can be expressed as the difference of $\check{p}^{-1} \otimes \check{q}$ from $\check{1}$. But minimizing the distance of a quaternion from the unit quaternion is the same as minimizing the rotation angle Ψ which is, in turn, the same as maximizing the zeroeth component of the quaternion, $\cos(\Psi/2)$. Note that, the quaternions $\check{p}^{-1} \otimes \check{q}$ and $-\check{p}^{-1} \otimes \check{q}$ affect the same rigid body rotation in 3-dimensional Euclidean space, but minimizing the distance of one from $\check{1}$ maximizes the distance of the other in quaternion space. So we just extremize the distance, rather than specifically minimize it. Once we have the solution set, we can check which solutions correspond to a maximum and which to a minimum, and choose the latter.

We therefore need to extremize the zeroeth component of $\check{p}^{-1}\check{q}$, where $\check{p} = [p_0 \ p_1 \ p_2 \ p_3]^T$ is the attitude estimate obtained by integrating the angular velocity ω as given in equation (13) and $\check{q}$ is expressed in terms of c/s and n_3 as in equation (9), while enforcing the constraint in equation (1). This can be accomplished by using the method of Lagrange multipliers to define a cost function that invokes the error norm as well as the constraint. Below, we have multiplied the cost function by the constant g_3 and the constraint by g_3^2, noting that the solution is unaffected by such a scaling:

$$J(\Phi, n_3) = g_3[\check{p}^{-1} \otimes \check{q}]_0 + \lambda g_3^2 (n_1^2 + n_2^2 + n_3^2 - 1) \\ = (cp_0 + sn_3 p_3) g_3 + (-cf_2 + sn_3 g_1) p_1 + (cf_1 + sn_3 g_2) p_2 \\ + \lambda \left(n_3^2 g^T g + 2\frac{cn_3}{s}(f_1 g_2 - f_2 g_1) + \frac{c^2}{s^2}(f_1^2 + f_2^2) - g_3^2 \right) \\ = c(g_3 p_0 + f_1 p_2 - f_2 p_1) + sn_3 g^T p \\ + \lambda \left(n_3^2 g^T g + 2\frac{cn_3}{s}(f_1 g_2 - f_2 g_1) + \frac{c^2}{s^2}(f_1^2 + f_2^2) - g_3^2 \right) , \tag{15}$$

where p denotes the vector portion of $\check{p}$. Now we set the first order partial derivatives of J to 0:

$$0 = \partial_\Phi J = -s(g_3 p_0 + f_1 p_2 - f_2 p_1) + cn_3 g^T p \\ + \left(-\frac{2\lambda}{s^2}\right)\left(\frac{c}{s}(f_1^2 + f_2^2) + n_3(f_1 g_2 - f_2 g_1)\right) \ , \tag{16}$$

$$0 = \partial_{n3} J = s g^T p + 2\lambda g^T g n_3 + 2\lambda \frac{c}{s}(f_1 g_2 - f_2 g_1) \ , \quad (17)$$

$$0 = \partial_\lambda J = n_3^2 g^T g - g_3^2 + \frac{2cn_3}{s}(f_1 g_2 - f_2 g_1) + \frac{c^2}{s^2}(f_1^2 + f_1^2) \quad (18)$$

Equation (17) yields:

$$-2\lambda = \frac{s g^T p}{g^T g n_3 + \frac{c}{s}(f_1 g_2 - f_2 g_1)} \ . \quad (19)$$

Substituting this in equation (16), we obtain:

$$\begin{aligned} &- s(g_3 p_0 + f_1 p_2 - f_2 p_1) \left(g^T g n_3 + \frac{c}{s}(f_1 g_2 - f_2 g_1) \right) + \\ &g^T p \left[c n_3 \left(g^T g n_3 + \frac{c}{s}(f_1 g_2 - f_2 g_1) \right) \right. \\ &\left. + \frac{c}{s^2}(f_1^2 + f_2^2) + \frac{n_3}{s}(f_1 g_2 - f_2 g_1) \right] = 0 \ . \end{aligned} \quad (20)$$

The factor in the square brackets can be substantially simplified using the constraint equation (18) as:

$$\begin{aligned} &c n_3 \left(g^T g n_3 + \frac{c}{s}(f_1 g_2 - f_2 g_1) \right) + \frac{c}{s^2}(f_1^2 + f_2^2) \\ &\quad + \frac{n_3}{s}(f_1 g_2 - f_2 g_1) \\ &= c \left(g_3^2 - \frac{c^2}{s^2}(f_1^2 + f_2^2) - \frac{c n_3}{s}(f_1 g_2 - f_2 g_1) \right) \\ &\quad + \frac{c}{s^2}(f_1^2 + f_2^2) + \frac{n_3}{s}(f_1 g_2 - f_2 g_1) \\ &= c(g_3^2 + f_1^2 + f_2^2) + s n_3 (f_1 g_2 - f_2 g_1) \ . \end{aligned}$$

Substituting this back into equation (20), we obtain:

$$\begin{aligned} &- (g_3 p_0 + f_1 p_2 - f_2 p_1) \left(s g^T g n_3 + c(f_1 g_2 - f_2 g_1) \right) \\ &\quad + g^T p (c(g_3^2 + f_1^2 + f_2^2) + s n_3 (f_1 g_2 - f_2 g_1)) = 0 \ . \end{aligned}$$

Accumulating terms containing sn_3 and c, we obtain an expression for the ratio $\kappa = c/(sn_3)$ in terms of known quantities as:

$$\kappa = \frac{(g_3 p_0 + f_1 p_2 - f_2 p_1) g^T g - g^T p (f_1 g_2 - f_2 g_1)}{g^T p (f_1^2 + f_2^2 + g_3^2) - (g_3 p_0 + f_1 p_2 - f_2 p_1)(f_1 g_2 - f_2 g_1)} \ , \quad (21)$$

where $f = h - b$ and $g = h + b$ were defined in terms of the vector measurements, and $\check{p}$ is obtained by integrating the angular velocities. Fortuitouosly, $c/s = \cot(\Phi/2)$ is therefore just proportional to n_3, and upon expressing c/s in terms of n_3 in the normalization constraint (equation (18)), the resulting equation becomes extremely simple to solve:

$$g_3^2 = g^T g n_3^2 + 2\kappa(f_1 g_2 - f_2 g_1) n_3^2 + \kappa^2 n_3^2 (f_1^2 + f_2^2) \ ,$$

or

$$n_3 = \frac{g_3}{\sqrt{g^T g + 2\kappa(f_1 g_2 - f_2 g_1) + \kappa^2 (f_1^2 + f_2^2)}} \ , \quad (22)$$

$$\frac{c}{s} = \frac{\kappa g_3}{\sqrt{g^T g + 2\kappa(f_1 g_2 - f_2 g_1) + \kappa^2 (f_1^2 + f_2^2)}} \ . \quad (23)$$

The other components of the attitude quaternion can be obtained using the inner two rows of equation (9). Thus we obtain the following theorem.

Theorem 5. *If the angular velocity of a rigid body is integrated to yield a attitude quaternion estimate $\check{p}$, then the estimate $\check{q} \in Q_b$ lying in the feasibility cone of measurement b which is closest to $\check{p}$, is given by equations (9, 21, 22, 23).*

Proof. The proof follows from the construction leading to equations (9, 21, 22, 23). Refer figure 2. □

Remark 5.1. *Sign indeterminacy*: There are two instances of taking square-roots in the construction of the optimal estimate: one in the denominators in equations (22, 23), and a second when determining $s = 1/\sqrt{(c/s)^2 + 1}$. They multiply all the components, and thus result in a net sign indeterminacy of the complete quaternion. We could choose the sign as yielded by the equations, or such that the zeroeth component is positive. Both choices yield a correct attitude in three-dimensional Euclidean space.

Remark 5.2. *Solution when reference vector is aligned with z-axis*: A common application of the presented solution would be to an aerial robot that uses an accelerometer to measure the gravity vector (after acceleration compensation). Since the reference coordinate system's z-axis is aligned with the reference vector $\mathbf{h}$, we have $f = [(-b_1)\ (-b_2)\ (1 - b_3)]^T$ and $g = [b_1\ b_2\ (1 + b_3)]^T$. Equations (21, 23) now simplify to:

$$\kappa = \frac{c}{s n_3} = \frac{(1 + b_3) p_0 - b_1 p_2 + b_2 p_1}{b_1 p_1 + b_2 p_2 + (1 + b_3) p_3} \ ,$$

$$\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} c \\ s n_1 \\ s n_2 \\ s n_3 \end{bmatrix} = \frac{1}{\sqrt{2(1 + \kappa^2)(1 + b_3)}} \begin{bmatrix} \kappa(1 + b_3) \\ \kappa b_2 + b_1 \\ -\kappa b_1 + b_2 \\ (1 + b_3) \end{bmatrix} \ , \quad (24)$$

where we have used the fact that $(1+b_3)^2 + b_1^2 + b_2^2 = 2(1+b_3)$. While the introduction of the auxillary variable κ in equations (21 - 23) seems adhoc, its role is more clearly visible now – κ parameterizes the feasibility cone Q_b in terms of the two special solutions provided in lemma 2:

$$\sqrt{2(1 + \kappa^2)(1 + b_3)} \check{q} = \kappa \begin{bmatrix} 1 + b_3 \\ b_2 \\ -b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_1 \\ b_2 \\ 1 + b_3 \end{bmatrix} \ ,$$

$$\text{or,} \qquad \check{q} = (\kappa \check{r} + \check{s})/\sqrt{1 + \kappa^2} \ . \quad (25)$$

Equation (24) may be checked for sanity against the Euler angle solution by using the relations $\sin\theta = 2(q_0 q_2 - q_1 q_3)$, $\cos\theta \sin\phi = 2(q_0 q_1 + q_2 q_3)$, and $\cos\theta\cos\phi = q_0^2 - q_1^2 - q_2^2 + q_3^3$. The reduction of the quaternion form to the Euler angle form is straightforward, but the details are long and omitted. The final result is that

$$\begin{bmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta\cos\phi \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \ .$$

So,

$$\begin{bmatrix} \tan\phi \\ \sin\theta \end{bmatrix} = \begin{bmatrix} b_2/b_3 \\ -b_1 \end{bmatrix} \ ,$$

as expected.

Remark 5.3. *Relation to the EKF*: A filtered attitude estimate $\check{q}_f$ can be obtained by suitable interpolation between the integrated estimate $\check{p}$ and the vector aligned estimate $\check{q}$ of theorem 5:

$$\check{q}_f = \check{p} \otimes (\check{p}^{-1} \otimes \check{q})^x . \tag{26}$$

The optimal interpolation parameter x in equation 6 is obtained from the measurement noise variances, similar to the Kalman gain in an EKF. An exact expression for x under a standard set of assumptions is derived in the following subsection. In the limit $\omega T \to 0$, the interpolated estimate is identical to the solution obtained using an optimally tuned EKF. However, for large incremental changes in the attitude, the geometric solution presented in this note is superior to the EKF which suffers from a loss of accuracy on account of the linearization.

C. Gyroscopic bias estimation

With the simplifying choice for the reference coordinate system's z-axis that leads to equation (24), the following corollary is apparent.

Corollary 6. *The correction that takes the integrated estimate $\check{p}$ into the feasibility cone Q_b is essentially a rotation about an axis that is orthogonal to the reference vector h.*

Proof. The correcting rotation in the reference coordinate system is:

$$\check{r} = \check{q} \otimes \check{p}^{-1} = \begin{bmatrix} \kappa(1+b_3) \\ \kappa b_2 + b_1 \\ -\kappa b_1 + b_2 \\ (1+b_3) \end{bmatrix} \otimes \begin{bmatrix} p_0 \\ -p_1 \\ -p_2 \\ -p_3 \end{bmatrix} / \sqrt{2(1+\kappa^2)(1+b_3)} \,.$$

So, using the expression for κ in equation (24), we obtain $r_3 = 0$. ☐

The underlying reason for this result is just that a rotation about any other axis would have an unnecessary component about h, and that would make the correction to reach Q_b suboptimal.

Theorem 7. *In the absence of any other measurement errors, a fixed bias error in the angular velocity measurement can be completely estimated by applying theorem 5 on two linearly independent vector measurements.*

Proof. Similar to the proof of corollary 6, the incremental change from the integrated attitude quaternion estimate, $\check{p}$, to the vector-aligned estimate, $\check{q}$, is essentially the correction to the integrated error in the rate measurement $e(t)$. Denoting the increment by $\check{r}$, now in the body-fixed coordinate system (since ω is available only in this system), for a constant e over a small integration time δt, we must have:

$$\check{r} = \begin{bmatrix} 1 \\ \delta r \end{bmatrix} = \check{p}^{-1} \otimes \check{q} = \begin{bmatrix} 1 \\ (e\delta t)/2 \end{bmatrix} + \delta\mu \check{b} \,, \tag{27}$$

where $\delta\mu$ is an unknown infinitesimal rotation about b in the body system. We have assumed that we start on the feasibility cone, and integrate the rate measurement over a small time, so $\check{r}$ is close to $\check{1}$, and its scalar portion is approximately 1. However, with a single vector measurement, a correction is possible only in the subspace normal to the measured vector b. Therefore, we have an unknown term proportional to $\check{b}$ in equation (27). Projecting onto the subspace orthogonal to $\check{b}$, we obtain $(1 - bb^T)e = 2(1 - bb^T)\delta r/\delta t$ in the case of a correction onto the feasibility cone of a single measurement b. Since b and δr are known, this may be used to estimate the portion of e normal to b. With two or more independent measurements b_i and corrections r_i at a constant e, the matrix $\sum_i (1 - b_i b_i^T)$ becomes invertible, and we can actually determine e completely:

$$\sum_i (1 - b_i b_i^T) e = \sum_i 2(1 - b_i b_i^T)\delta r_i/\delta t \,. \tag{28}$$

Equation (28) thus constructs the desired correction for the bias in the rate measurement. ☐

Remark 7.1. *Observability condition*: The condition for invertibility of $\sum_i (1 - b_i b_i^T)$ is the same as the full-rank condition in literature, and for a single vector observation, it is equivalent to the persistently non-parallel and sufficient excitation conditions.

Remark 7.2. *Non constant bias*: If only measurements of a single constant vector are available, the body would have to rotate faster than the variation in e, if any such variation exists, for this estimation to be accurate. If e does happen to vary significantly, we would only be estimating the weighted average of the error, $\bar{e}$, during the time over which the measurements were taken and the corrections determined:

$$\sum_i (1 - b_i b_i^T) \bar{e} = \sum_i 2(1 - b_i b_i^T)\delta r_i/\delta t \,. \tag{29}$$

Remark 7.3. *Random-walk bias*: Filtered bias estimation for variable bias with exponential autocorrelation (WIP). Filtering details to be expanded.

$$\begin{aligned} A_{i+1} &= (1 - T/\tau)A_i + (1 - b_i b_i^T) \\ B_{i+1} &= (1 - T/\tau)B_i + 2(1 - b_i b_i^T)\delta r_i/\delta t \\ A\bar{e} &= B. \end{aligned} \tag{30}$$

D. Effect of noise in measurements on the estimation

We now analyze the effect of random, unbiased noise in the angular velocity measurement ω and vector measurement b on the estimated attitude $\check{q}$. In particular, we shall assume that there is no bias error in ω. Further, we shall make the reasonable assumption that the errors are small enough relative to the norms of the quantities to consider them as perturbations, and therefore add the effects of individual noise sources to obtain the cumulative effect.

We shall introduce some new notation, to avoid lengthy expressions. The quaternion attitude estimate is given by equation (24):

$$\sqrt{2(\alpha^2+\beta^2)(1+b_3)}\, \check{q} = \begin{bmatrix} \alpha(1+b_3) \\ \alpha b_2 + \beta b_1 \\ -\alpha b_1 + \beta b_2 \\ \beta(1+b_3) \end{bmatrix} = \alpha \check{u} + \beta \check{v} \,, \tag{31}$$

where $\check{u} = [(1+b_3)\ b_2\ -b_1\ 0]^T$ and $\check{v} = [0\ b_1\ b_2\ (1+b_3)]^T$ are scaled versions of the two special solutions from lemma 2, $\alpha = p_0(1+b_3) + p_1 b_2 - p_2 b_1 = \check{p}^T \check{u}$, and $\beta = p_1 b_1 + p_2 b_2 + p_3(1+b_3) = \check{p}^T \check{v}$.

Let us first consider the effect of noise in ω alone. Suppose the noise in ω leads to a small error in the integrated estimate $\delta\check{p} = (T/2)\check{q} \otimes \delta\check{\omega}$ (refer equation (14) for a small T). The errors in α, β would then be:

$$\begin{bmatrix}\delta\alpha \\ \delta\beta\end{bmatrix} = \begin{bmatrix} 1+b_3 & b_2 & -b_1 & \\ & b_1 & b_2 & 1+b_3 \end{bmatrix} \delta\check{p} = \begin{bmatrix}\check{u}^T \\ \check{v}^T\end{bmatrix} \delta\check{p}\,.$$

Theorem 8. *In the absence of any other errors, a perturbation error $\delta\check{p}$ in the integrated attitude estimate $\check{p}$ leads to a perturbation in the vector-aligned attitude estimate $\check{q}$ (equation (24)) equal to the projection of $\delta\check{p}$ onto the feasibility cone, i.e., the subspace spanned by the two special solutions in lemma 2, and orthogonal to the nominal attitude estimate.*

Proof. Taking differentials of equation (31):

$$\sqrt{2(\alpha^2+\beta^2)(1+b_3)}\,\delta\check{q} = \left\{ \begin{array}{r} -\dfrac{(\alpha\delta\alpha + \beta\delta\beta)\sqrt{2(1+b_3)}}{\sqrt{\alpha^2+\beta^2}}\check{q} \\ +\check{u}\delta\alpha + \check{v}\delta\beta \end{array} \right. ,$$

$$= (1 - \sqrt{\frac{2(1+b_3)}{\alpha^2+\beta^2}}\check{q}\check{p}^T)(\check{u}\delta\alpha + \check{v}\delta\beta)\,. \qquad (32)$$

Once we have expressed the error as the sum of first order differentials, the multiplying coefficients may now be approximated to their nominal values – any error on account of the approximation would be multiplied by the differentials and therefore be of higher order. Specifically, we may approximate $\check{p} \approx \check{q}$, so $\check{p}\otimes\check{b} \approx \check{h}\otimes\check{p}$, so $\alpha \approx 2p_0 \approx 2q_0$, and $\beta \approx 2p_3 \approx 2q_3$, in the coefficients, to obtain

$$\begin{aligned} \alpha^2+\beta^2 &= 4q_0^2 + 4q_3^3 = 2(1+b_3)\,, \\ 2(1+b_3)\delta\check{q} &= (1-\check{q}\check{q}^T)\begin{bmatrix}\check{u} & \check{v}\end{bmatrix}\begin{bmatrix}\delta\alpha \\ \delta\alpha\end{bmatrix}\,, \\ &= (1-\check{q}\check{q}^T)\begin{bmatrix}\check{u} & \check{v}\end{bmatrix}\begin{bmatrix}\check{u}^T \\ \check{v}^T\end{bmatrix}\delta\check{p}\,, \\ &= (1-\check{q}\check{q}^T)(\check{u}\check{u}^T + \check{v}\check{v}^T)\delta\check{p} \\ \delta\check{q} &= (1-\check{q}\check{q}^T)(\check{r}\check{r}^T + \check{s}\check{s}^T)\delta\check{p} = \check{t}\check{t}^T\delta\check{p}\,, \qquad (33) \end{aligned}$$

where $\check{t} \in Q_b$, and $\check{t} = (-\check{r} + \kappa\check{s})/\sqrt{1+\kappa^2} = \check{h} \otimes \check{q}$. □

A similar but tedious derivation yields the following theorem for noise in the vector measurement b. We shall reuse some of the previous notation leading to theorem 8 and equation (31).

Theorem 9. *In the absence of any other errors, a perturbation error $\delta\check{b}$ in the vector measurement $\check{b}$ leads to a perturbation in the vector-aligned attitude estimate $\check{q}$ (equation (24)) equal to a rotation through the angle $-b\times\delta b$, which is the smallest angle rotation that takes b to $b+\delta b$.*

Proof. Taking differentials of equation (31):

$$\left. \begin{array}{r} \sqrt{2(\alpha^2+\beta^2)(1+b_3)}\delta\check{q} \\ +\check{q}\sqrt{2(1+b_3)}\dfrac{\alpha\delta\alpha+\beta\delta\beta}{\sqrt{\alpha^2+\beta^2}} \\ +\check{q}\sqrt{2(\alpha^2+\beta^2)}\dfrac{\delta b_3}{2\sqrt{1+b_3}} \end{array} \right\} = \left\{ \begin{array}{c} \delta\alpha\check{u} + \delta\beta\check{v} \\ +\alpha\delta\check{u} + \beta\delta\check{v} \end{array} \right. . \qquad (34)$$

Similar to the proof of theorem 8, the coefficients multiplying the first order differentials are approximated to their nominal values.

$$\begin{aligned} \alpha^2+\beta^2 &= 4q_0^2 + 4q_3^2 = 2(1+b_3)\,, \\ \begin{bmatrix}\alpha & \beta\end{bmatrix}\begin{bmatrix}\delta\alpha \\ \delta\beta\end{bmatrix} &= \check{p}^T\begin{bmatrix}\check{u} & \check{v}\end{bmatrix}\begin{bmatrix}\delta\alpha \\ \delta\beta\end{bmatrix} = \check{q}^T\begin{bmatrix}\check{u} & \check{v}\end{bmatrix}\begin{bmatrix}\delta\alpha \\ \delta\beta\end{bmatrix}\,. \qquad (35) \end{aligned}$$

Working on the $\delta\check{u}$ and $\delta\check{v}$ terms,

$$\begin{aligned} \check{q}^T(\alpha\delta\check{u} + \beta\delta\check{v}) &= \check{q}^T(2q_0\delta\check{u} + 2q_3\delta\check{v})\,, \\ &= 2\check{q}^T\left\{ q_0\begin{bmatrix} & & 1 \\ & 1 & \\ -1 & & \\ & & \end{bmatrix} + q_3\begin{bmatrix} & & \\ 1 & & \\ & 1 & \\ & & 1\end{bmatrix}\right\}\delta b\,, \\ &= 2(q_0\begin{bmatrix}-q_2 & q_1 & q_0\end{bmatrix} + q_3\begin{bmatrix}q_1 & q_2 & q_3\end{bmatrix})\delta b\,, \\ &= \begin{bmatrix}b_1 & b_2 & (1+b_3)\end{bmatrix}\delta b = \delta b_3\,. \qquad (36) \end{aligned}$$

Substituting from equations (35, 36) back in equation (34),

$$\left. \begin{array}{r} 2(1+b_3)\delta\check{q} + \check{q}\check{q}^T(\check{u}\delta\alpha + \check{v}\delta\beta) \\ +\check{q}\check{q}^T(\alpha\delta\check{u} + \beta\delta\check{v}) \end{array} \right\} = \left\{ \begin{array}{l} \check{u}\delta\alpha + \check{v}\delta\beta \\ \ +\alpha\delta\check{u} + \beta\delta\check{v} \end{array} \right. ,$$

It can be seen that the terms on the RHS are projected onto $\check{q}$ and the projection appears on the LHS. This is just a consequence of the fact that $\check{q}$ has unit magnitude, and therefore $\delta\check{q}$ must be orthogonal to $\check{q}$:

$$2(1+b_3)\delta\check{q} = (1-\check{q}\check{q}^T)\left(\check{u}\delta\alpha + \check{v}\delta\beta + \alpha\delta\check{u} + \beta\delta\check{v}\right)\,. \qquad (37)$$

We now simplify the terms within the parantheses on the RHS using the relations that $b^T\delta b = 0$ and $\check{q}\otimes\check{b} = \check{h}\otimes\check{q}$:

$$\begin{aligned} &\check{u}\delta\alpha + \check{v}\delta\beta + \alpha\delta\check{u} + \beta\delta\check{v} \\ &= \begin{bmatrix} -q_2(1+b_3) & q_1(1+b_3) & q_0(1+b_3) \\ (-q_2b_2 + q_1b_1) & (q_1b_2 + q_2b_1) & (q_0b_2 + q_3b_1) \\ (q_2b_1 + q_1b_2) & (-q_1b_1 + q_2b_2) & (-q_0b_1 + q_3b_2) \\ q_1(1+b_3) & q_2(1+b_3) & q_3(1+b_3) \end{bmatrix}\delta b \\ &\quad + 2q_0\begin{bmatrix} & & 1 \\ & 1 & \\ -1 & & \\ & & \end{bmatrix}\delta b + 2q_3\begin{bmatrix} & & \\ 1 & & \\ & 1 & \\ & & 1\end{bmatrix}\delta b\,, \\ &= \begin{bmatrix} -q_2(1+b_3) & q_1(1+b_3) & 0 \\ (2q_3 - q_2b_2 + q_1b_1) & (2q_0 + q_1b_2 + q_2b_1) & 0 \\ (-2q_0 + q_2b_1 + q_1b_2) & (2q_3 - q_1b_1 + q_2b_2) & 0 \\ q_1(1+b_3) & q_2(1+b_3) & 0 \end{bmatrix}\delta b \\ &\quad + \begin{bmatrix} 0 & 0 & q_0(1+b_3) \\ 0 & 0 & (q_0b_2 + q_3b_1) \\ 0 & 0 & (-q_0b_1 + q_3b_2) \\ 0 & 0 & q_3(1+b_3) \end{bmatrix}\delta b + 2\begin{bmatrix} q_0 \\ \\ \\ q_3 \end{bmatrix}\delta b_3\,, \end{aligned}$$

(Using $\check{q}\otimes\check{b} = \check{h}\otimes\check{q}$)

$$\begin{aligned} &= \begin{bmatrix} -q_2(1+b_3) & q_1(1+b_3) & 0 \\ (2q_1b_1 + q_3(1+b_3)) & (2q_1b_2 + q_0(1+b_3)) & 0 \\ (2q_2b_1 - q_0(1+b_3)) & (2q_2b_2 + q_3(1+b_3)) & 0 \\ q_1(1+b_3) & q_2(1+b_3) & 0 \end{bmatrix}\delta b \\ &\quad + \begin{bmatrix} 0 & 0 & q_0(1+b_3) \\ 0 & 0 & (q_0b_2 + q_3b_1) \\ 0 & 0 & (-q_0b_1 + q_3b_2) \\ 0 & 0 & q_3(1+b_3) \end{bmatrix}\delta b + 2\begin{bmatrix} q_0 \\ \\ \\ q_3 \end{bmatrix}\delta b_3\,, \end{aligned}$$

(Using $b^T \delta b = 0$)

$$= (1+b_3)\begin{bmatrix} -q_2 & q_1 & q_0 \\ q_3 & q_0 & -q_1 \\ -q_0 & q_3 & -q_2 \\ q_1 & q_2 & q_3 \end{bmatrix} \delta b + \begin{bmatrix} 0 \\ q_1 - q_1 b_3 + q_0 b_2 + q_3 b_1 \\ q_2 - q_2 b_3 - q_0 b_1 + q_3 b_2 \\ 0 \end{bmatrix} \delta b_3 + 2 \begin{bmatrix} q_0 \\ \\ \\ q_3 \end{bmatrix} \delta b_3 \,,$$

(Again using $\check{q} \otimes \check{b} = \check{h} \otimes \check{q}$)

$$= (1+b_3)\begin{bmatrix} -q_2 & q_1 & q_0 \\ q_3 & q_0 & -q_1 \\ -q_0 & q_3 & -q_2 \\ q_1 & q_2 & q_3 \end{bmatrix} \delta b + 2 \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \delta b_3 \,,$$

$$= (1+b_3)\begin{bmatrix} -q_2 & q_1 & q_0 \\ q_3 & q_0 & -q_1 \\ -q_0 & q_3 & -q_2 \\ q_1 & q_2 & q_3 \end{bmatrix} \delta b + 2\check{q}\delta b_3 \,. \tag{38}$$

Substituting from equation (38) back in equation (37), we obtain:

$$\delta\check{q} = -\frac{1}{2}\check{q} \otimes \check{b} \otimes \delta\check{b} = -\frac{1}{2}\check{t} \otimes \delta\check{b} \,, \tag{39}$$

where $\check{t} = \check{h} \otimes \check{q} = [(-q_3)\ (-q_2)\ q_1\ q_0]^T = \check{q} \otimes \check{b}$, and $\check{q} \otimes \check{b}$ is already orthogonal to $\check{q}$. □

A quick consistency check may be obtained using equation 1:

$$\check{q} \otimes \check{b} = \check{h} \otimes \check{q},$$
$$\Rightarrow \delta\check{q} \otimes \check{b} + \check{q} \otimes \delta\check{b} = \check{h} \otimes \delta\check{q}.$$

Checking equation 39,

$$-(1/2)\check{q} \otimes \check{b} \otimes \delta\check{b} \otimes \check{b} + \check{q} \otimes \delta\check{b} + (1/2)\check{h} \otimes \check{q} \otimes \check{b} \otimes \delta\check{b} \stackrel{?}{=} 0,$$
$$\Leftarrow (1/2)\check{q} \otimes \check{b} \otimes \check{b} \otimes \delta\check{b} + \check{q} \otimes \delta\check{b} + (1/2)\check{q} \otimes \check{b} \otimes \check{b} \otimes \delta\check{b} \stackrel{?}{=} 0,$$
$$\Leftarrow -(1/2)\check{q} \otimes \delta\check{b} + \check{q} \otimes \delta\check{b} - (1/2)\check{q} \otimes \delta\check{b} \stackrel{?}{=} 0. \checkmark$$

Equations (14, 33, 39) can be used to derive an equation for the evolution of noise in the integrated and vector-aligned estimates:

$$\delta\check{p}_{i+1} = \delta\check{q}_i \otimes \left(\check{1} + \frac{\check{\omega}_i T}{2}\right) + \check{q}_i \otimes \frac{\delta\check{\omega}_i T}{2} = P \begin{bmatrix} \delta\check{q}_i \\ \delta\omega_i \end{bmatrix},$$
$$\delta_b\check{q} = -(1/2)\check{q} \otimes \check{b} \otimes \delta\check{b} = -(1/2)\check{h} \otimes \check{q} \otimes \delta\check{b},$$
$$\delta_p\check{q} = \check{t}\check{t}^T\delta\check{p},$$
$$\Rightarrow \delta\check{q}_{i+1} = \check{t}_{i+1}\check{t}_{i+1}^T \left[\delta\check{q}_i \otimes \left(\check{1} + \frac{\check{\omega}_i T}{2}\right) + \check{q}_i \otimes \frac{\delta\check{\omega}_i T}{2}\right]$$
$$- \frac{1}{2}\check{t}_{i+1} \otimes \delta\check{b}_{i+1} = Q \begin{bmatrix} \delta\check{q}_i \\ \delta\omega_i \\ \delta b_i \end{bmatrix}. \tag{40}$$

Equation (40) can be used to derive expressions for the covariance matrices corresponding to $\check{p}_{i+1}$ and $\check{q}_{i+1}$, say Π_{i+1} and Ξ_{i+1}, which can be used in deriving a filtered estimate:

$$\delta\check{p}_{i+1} = P \begin{bmatrix} \delta\check{q}_i \\ \delta\omega_i \end{bmatrix},$$
$$\delta\check{q}_{i+1} = Q \begin{bmatrix} \delta\check{q}_i \\ \delta\omega_i \\ \delta b_{i+1} \end{bmatrix},$$
$$\Pi_{i+1} = P \begin{bmatrix} \Xi_i & \\ & W_i \end{bmatrix} P^T,$$
$$\Xi_{i+1} = Q \begin{bmatrix} \Xi_i & & \\ & W_i & \\ & & B_{i+1} \end{bmatrix} Q^T,$$
$$\check{q}_{f,i+1} = (\Pi_{i+1}^{-1} + \Xi_{i+1}^{-1})^{-1}(\Pi_{i+1}^{-1}\check{p}_{i+1} + \Xi_{i+1}^{-1}\check{q}_{i+1}), \tag{41}$$

where W and B are the covariance matrices corresponding to measurements ω and b.

IV. Simulation results

In this section, we use Matlab simulations to verify the key theoretical results derived in the previous section. The first group of simulations correspond to verifying the solution for the first problem – attitude estimation using two vector measurements. We assume that the directions of two linearly independent vectors, h and k, are measured at 100Hz in the body-fixed coordinate system as a and b. Measurements a and b are assumed to have random, unbiased noise of 0.01 and 0.02 normalized units respectively. The body is prescribed an oscillatory roll and pitch motion, and a constant yaw angle.

Figure 3 (left) shows the attitude estimated using theorem 4, $\check{q}_G$, in comparison with the attitude derived by using the TRIAD method, $\check{q}_T$, when reference vector h is of greater significance. Both the solutions are identical upto machine precision.

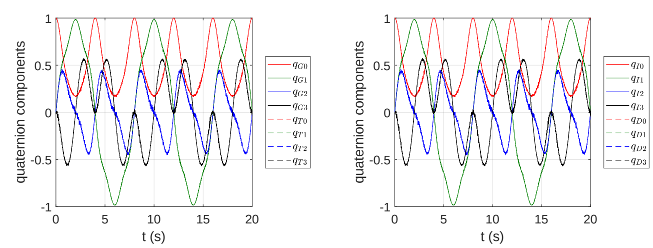

Fig. 3. Matlab simulations of full attitude estimation using two vector measurements. The dashed lines on the left figure correspond to the TRIAD solution, while those on the right figure correspond to Davenport's q-method. The solid lines in both figures correspond to the attitude estimated using the methods presented in this note, *viz*, theorem 4 and equation (6). The solid lines match the dashed lines to machine precision.

By using equation (6) to interpolate between the two solutions obtained from theorem 4, we obtain the solution to Wahba's problem. The interpolation parameter x is chosen to be $2^2/(1^2 + 2^2) = 0.8$, as the noise of the two vector measurements have a ratio of 2. Figure 3 (right) shows the equivalence between the result obtained by interpolating (equation (6)) on the two estimates of theorem 4, $\check{q}_I$, and that obtained by using Davenport's q-method, $\check{q}_D$.

The next group of simulations verify the result of theorem 5. In these simulations, we assume a constant gyroscopic bias of $[-0.32\ 0.16\ -0.08]^T$rad/s along the three axes, and a random, unbiased noise of 0.04rad/s in each component. The reference vector components are assumed to be $h = [0\ 0\ 1]^T$. The vector measurement is also assumed to have a random, unbiased noise of 0.01 normalized units, but we assume any constant biases in this measurement have been eliminated. The vector measurement is then normalized before being passed on to the attitude estimator.

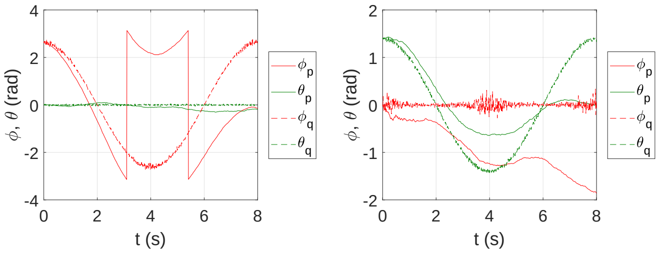

Fig. 4. Simulated attitude estimation for pure sinusoidal roll (left) and pitch (right) manoeuvres. While the gyro integrated estimate drifts with time, the vector measurement correction (equation (24)) realigns the roll and pitch angles at every time step.

The quaternion output of the attitude estimator is converted to 3-2-1 Euler angles for ease of readability. The angular velocity is measured at 100Hz and integrated (along with the bias and noise errors) to return an integrated estimate for the attitude (ϕ_p and θ_p after conversion to roll and pitch Euler angles). Then, a corrected attitude is determined that is consistent with the noisy vector measurement, also at 100Hz, to yield the vector-aligned estimate (ϕ_q and θ_q respectively).

In this case of the reference vector being aligned with the z-axis, the attitude estimator cannot correct for errors on account of yaw drift in the integrated estimate $\check{p}$. Therefore, we can evaluate the estimator's performance after isolating the roll and pitch angles from the estimate. The first plot (figure 4 left) considers the case of a sinusoidal roll manoeuvre of amplitude $\pm 5\pi/6$rad and frequency 0.25Hz. The second plot (figure 4 right) repeats the simulation with a fixed roll angle and a sinusoidal pitch manoeuvre of amplitude $\pm 4\pi/9$rad and frequency 0.25Hz. It can be seen that the integrated estimates drift with time, but the vector-aligned estimates, while having more noise, stay true to the actual values.

The attitude estimate $\check{q}$ of theorem 5 can be filtered to reduce the noise variance using interpolation, as decribed in equation 26. For small attitude increments between time-steps, the filtered estimate is the same as that obtained using the traditional EKF, but the linearization inherent in the EKF begins to introduce significant errors for large attitude increments (figure 5).

We finally verify the estimation of a constant bias error in the angular velocity measurement as given in theorem 7, equation (28). This estimator requires the correction quaternion determined using atleast two linearly independent vector measurements, so the estimator is enabled only after the smallest eigenvalue of $\sum_i(1 - b_i b_i^T)$ is above a constant. Figure 6 (left) shows the simulation results that validate equation (28). As mentioned in the statement of theorem 7, the bias errors are assumed to be constant with time in this simulation. The estimated bias is then compensated in the attitude estimate (figure 6 right) in order to overcome the steady yaw drift with time. The initial error in the yaw attitude, and a random-walk error on account of the noise in ω are all that remain.

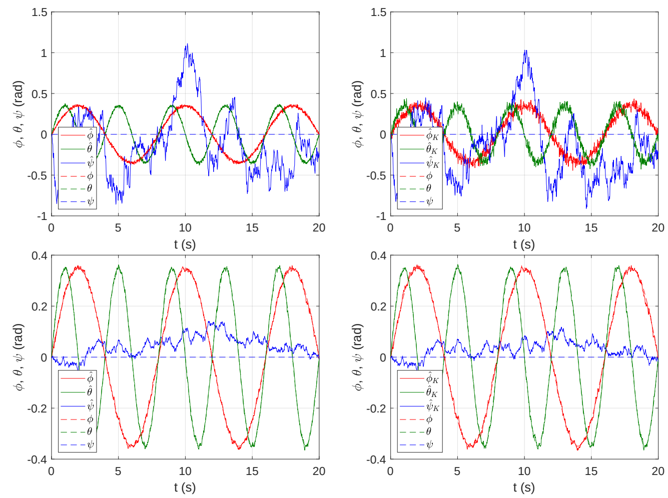

Fig. 5. Interpolation using equation (26) to obtain a filtered attitude estimate. The interpolated solution (top left) has lower errors than an optimally tuned EKF (top right) for large attitude increments (≈ 0.04 rad) between time-steps. In the limit of smaller attitude increments (≈ 0.004 rad), the EKF (bottom right) approaches the more accurate interpolated solution of equation (26) (bottom left).

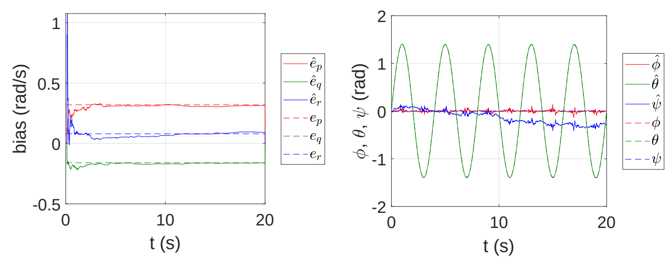

Fig. 6. Estimation of constant bias errors in the rate measurement from the correction required to align the integrated estimate with a vector measurement (solid lines on left). The bias error estimation uses equation (28) on attitude estimations derived from multiple linearly independent vector measurements. The estimated bias is utilized to correct for yaw drift (solid lines on right). The correction removes the yaw drift proportional to t, and only the initial error and a random-walk drift that goes as $\sqrt{t}$ remain.

V. Conclusion

We have thus reported a geometry-based analytic solution for the problem of attitude estimation using two reference vector measurements and using a rate measurement and a measurement of a single reference vector. The estimated attitude is exact in the sense that it yields the vector measurement exactly when applied on the reference measurement. The estimate also has no latency and is available at the same timestep when the measurement is available. The estimator is verified using Matlab simulations. It is also shown how the corrections to the integrated estimate can be used to estimate the time-averaged bias errors in the rate measurement. A perturbation analysis derives the effect of small signal noise in the measurements on the estimation, and can be used to derive a filtered attitude estimate using quaternion interpolation.

A concluding remark is that the presented approach can be extended in principle to problems involving more than two vector measurements. For instance, with three measurements, the problem reduces to one of determining the centroid with respect to three feasibility cones. On account of the nonlinear nature of the problem, it is not easy to obtain closed-form solutions to such problems by extending the presented solutions. Notwithstanding the algebraic difficulties involved, the conceptual extension to such problems is straightforward.

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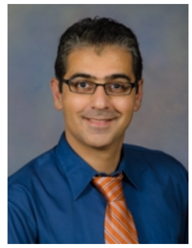

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