A unified geometric framework for rigid body attitude estimation ⋆

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Abstract

This paper presents solutions to the following two common quaternion attitude estimation problems: (i) estimation of attitude using measurement of two reference vectors, and (ii) estimation of attitude using rate measurement and measurement of a single reference vector. Both of these problems yield to a direct geometric analysis and solution. The former problem already has a well established analytic solution in literature using linear algebraic methods. This paper shows how the solution may also be obtained using geometric methods, which are not only more intuitive, but also amenable to unconventional extensions beyond the traditional least-squares formulations. With respect to the latter problem, existing solutions typically involve filters and observers and use a mix of differential-geometric and control systems methods. Again, this solution may also be derived analytically using the geometric method, which helps improve the estimation accuracy. In this paper, both the problems are formulated as angle optimization problems, which can be solved to obtain a unique closed-form solution. The proposed approach has the favourable consequences that the estimation is (i) exact, thus overcoming errors in solutions based upon linear methods, (ii) instantaneous with respect to the measurements, thus overcoming the latency inherent in solutions based upon negative feedback upon an error, which can at best show asymptotic convergence, and (iii) geometry-based, thus enabling imposition of geometric inequality constraints. The geometric approach has been verified in simulations as well as experiments, and its performance compared against existing methods.

Key words: Attitude estimation, geometric methods, quaternions, sensor fusion, nonlinear observers and filters.

1 Introduction

The problem of estimating the attitude of a rigid body with respect to a reference coordinate system, by measuring reference vectors in a body-fixed frame, has been treated abundantly in literature. One of the earliest, and arguably simplest, solution was Black's three-axis attitude estimator TRIAD [8]. A least squares formulation of the attitude estimation problem was posed by Wahba in [7]. Multiple solutions have been reported for Wahba's problem: using polar decomposition [12], an SVD method, Davenport's q-method [11], the Quaternion estimator QUEST [6], a factored-quaternion algorithm FQA [27], *etc.*

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Although both Davenport's q-method and QUEST use the quaternion representation of attitude, they ultimately reduce to an eigenvalue-eigenvector problem. Thus it can be seen that most solutions are linear algebraic in nature, and given the vast array of tools available for linear problems, they are all readily solved. This advantage is, however, associated with the accompanying weakness that it is not straightforward to incorporate nonlinear and nonholonomic constraints in the problem. For instance, in [14], the authors describe the attitude control of a spaceshuttle during a docking operation, when there is a hard constraint with respect to a nominal pitch angle in order to ensure that a trajectory control sensor is oriented towards the target platform. The attitude guidance module then estimates an optimal pitch attitude that complies with the hard constraint and minimizes the control effort. Similarly, in [25], the authors describe a reference governor with a pointing inclusion constraint such that the spacecraft points towards a fixed target, or an exclusion constraint such that sensitive equipment is not exposed to direct solar radiation. Such inequality constraints are obviously nonholonomic, and while being quite common in practice, are notoriously difficult to incorporate in a linear algebraic solution. Once the guidance or reference module determines an attitude that complies with the constraints, a controller module is used to achieve bounded or asymptotic stability with respect to the reference.

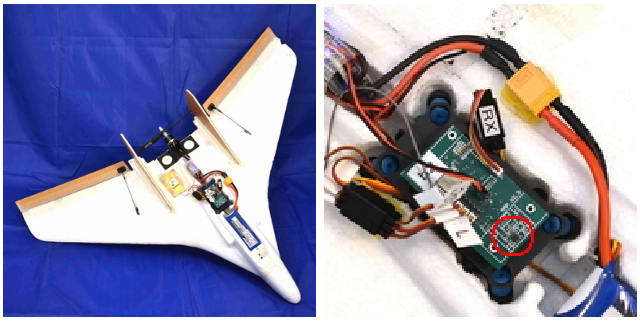

Fig. 1. Attitude estimation is a key engineering problem in the control of unmanned aeriel vehicles. Left: A 0.94m 0.66kg delta-wing UAV with an autopilot on board. Right: A zoomed in image of the autopilot [1] from Bingler and Mohseni, with the inertial sensors (the MPU9250 in the red circle).

Relatedly, the advent of small unmanned vehicles has motivated the development of solutions that depend upon minimal measurement resources in order to reduce the weight and cost of the sensor payload. In particular, it is of considerable interest to estimate the attitude using a single vector measurement, possibly supplemented by a rate measurement, thus leading us to the second of the stated problems. This interest is partly fueled by the availability of cheap commercial-off-the-shelf inertial measurement units (IMUs) that contain MEMS-based gyroscopes and accelerometers [24]. The research is also partly fueled by the realization that attitude estimation and control is a key challenge in the design of small autonomous aerial robots.

The second problem is most frequently solved using an extended Kalman filter (EKF) [5] or one of its several variants [17], [2]. The EKF provides a point-wise attitude estimate and is instantaneous with respect to the measurements. However, resulting from linearization of an intrinsically nonlinear problem, this solution is not robust to large changes in the attitude state [10].

More recently, some solutions have been reported in literature which use nonlinear observers or filters to solve the single-vector measurement problem [10], [3], [21], [9], [18]. These solutions have typically used an appropriate error signal in negative feedback to estimate the attitude. The solutions in [10], and [21] are quite general, and while having been developed for multiple vector measurements, they extend smoothly to the case of a single vector measurement. The solutions presented in [9], and [18] are more specific to the availability of single vector measurements. A common characteristic in this group of solutions is the use of negative feedback from an error signal to estimate the attitude and an (a-priori) unknown gain, that needs to be tuned in order to achieve satisfactory estimator performance. Such a feedback-based estimator is bound to have a finite latency with respect to the input, and cannot instantaneously track abrupt or discontinuous changes in the measurements, and the convergence of the estimate to the true attitude is at best asymptotic. An algebraic solution has been presented in [20] in the specific case of a gravity vector measurement. However, the solution does not directly extend to arbitrary time-varying reference vectors.The final word in rigid body attitude estimation has not been spoken yet, as evidenced by recent articles such as [17], [16], [2] and [22].

In contrast to the linear algebraic and filter approaches available in literature, this paper analyzes the attitude estimation problems from a geometric perspective. In the process, we obtain solutions that overcome some of the shortcomings in the previous solutions. Firstly, being of a geometric nature, the solutions easily extend to problems involving geometric constraints, irrespective of whether they are holonomic equations or nonholonomic inequality constraints. Secondly, the analytic solutions provide an instantaneous estimate for the attitude which is consistent with respect to the vector measurement at every time step. Besides the mathematical elegance of having an analytic solution, this also has several applications in autonomous guidance, navigation, and control systems: it enables the deployment of frugal single-vector-measurement sensor-suites, and the zero-latency accuracy of the solution is useful in multiple-vector-measurement suites in overcoming sudden failures or intermittent losses in some of the components without leading to large transient errors that could potentially cause system breakdown.

A brief outline of the paper is as follows. We begin by introducing the geometric approach and formulating the stated problems in the language of mathematics in section 2. The next section, section 3.1, presents the solution to the first problem, and relates it to the existing solutions from literature. The next section, Section 3.2, solves the second problem and also provides results relating to the accuracy of the solution. A filtering method is introduced in section 4 to address the issue of measurement noise. This is followed by verification of the theory using simulations and experiment in sections 5 and 6.

2 Notation, definitions, and problem formulation

In this section, we describe the geometry associated with vector measurements and formulate the attitude estimation problems as well-posed mathematical problems.

The attitude of the rigid body with respect to a reference coordinate system shall be represented using a unit quaternion, denoted using an angle-like check accent, *e.g.* $\check{p} = [p_0\ p_1\ p_2\ p_3]^T$, $\check{q} = [q_0\ q_1\ q_2\ q_3]^T \ldots$, such that $\check{p}^T\check{p} = \check{q}^T\check{q} = \ldots = 1$, so $\check{p}, \check{q} \in \mathbb{S}^3$, the unit 3-sphere. The product of two quaternions $\check{p}$ and $\check{q}$ shall be denoted as $\check{p} \otimes \check{q}$. Let $p = [p_1\ p_2\ p_3]^T$, and $q = [q_1\ q_2\ q_3]^T$.

$$\check{p} \otimes \check{q} = \begin{bmatrix} p_0 \\ p \end{bmatrix} \otimes \begin{bmatrix} q_0 \\ q \end{bmatrix} = \begin{bmatrix} p_0 q_0 - p^T q \\ p_0 q + p q_0 + p \times q \end{bmatrix}. \quad (1)$$

An attitude quaternion $\check{q} = [q_0\ q^T]^T$ is related to the rotation matrix C as:

$$C = (q_0^2 - q^T q)1_{3\times 3} + 2qq^T + 2q_0[q\times], \quad (2)$$

where $1_{m\times n}$ is the $m \times n$ identity matrix, and $[q\times]p$ is the vector product $q \times p$. The axis-angle formalism is related to $\check{q}$ as:

$$\check{q} = \begin{bmatrix} \cos(\Phi/2) \\ \sin(\Phi/2)n \end{bmatrix}, \quad (3)$$

for a rotation through Φ about the axis n. The attitude quaternion follows the kinematic equation

$$\dot{\check{q}} = \frac{1}{2}\check{q} \otimes \check{\omega} = \frac{1}{2}[\check{q}\otimes]\check{\omega} = \frac{1}{2}[\otimes\check{\omega}]\check{q}, \quad (4)$$

where $\check{\omega} = [0\ \omega^T]^T$ is the angular velocity quaternion, with $\omega \in \mathbb{R}^3$. The symbols $[\check{q}\otimes]$ and $[\otimes\check{\omega}]$ denote the left and right quaternion multiplication matrices.

The quaternion formalism leads to an elegant division algebra for rotations by furnishing simple algebraic operations for inversion ($\check{q} = [q_0\ q^T]^T \Rightarrow \check{q}^{-1} = [q_0\ -q^T]^T$), the composition of sequential rotations as quaternion mutiplication, and interpolation between rotations as geometric interpolation.

A reference vector, denoted in bold as $\mathbf{h}$, $\mathbf{k}$, $\ldots$, shall be defined as a unit magnitude vector that points in a specified direction. Examples include the direction of fixed stars relative to the body, the Earth's magnetic field, gravitational field *etc.* The components of any such direction may be measured in any three-dimensional orthogonal coordinate system. In the context of our problems, two obvious choices for the coordinate system are the reference coordinate system (relative to which the rigid body's attitude is to be determined), and a coordinate system fixed in the body. We assume the availability of measurement apparatus to obtain the vector's components in a three-dimensional orthogonal coordinate system, h, k, $\ldots$, a, b, $\ldots \in \mathbb{S}^2 \subset \mathbb{R}^3$ in the reference and body-fixed frames.

2.1 Geometry of vector measurement

A rotation quaternion (or, for that matter, any rotation representation) has three scalar degrees of freedom. A body-referred measurement b of a reference vector has 3 scalar components, that are related to the reference measurement h, in terms of the rotation quaternion. However, we also know that the measurement would retain the magnitude of the vector, *i.e.*, $h^T h = b^T b = 1$, so there is one scalar degree of redundancy in our measurement b and only two scalar degrees of information. Reconciling with this redundancy, we can therefore isolate the quaternion from a three-dimensional set of possibilities to a single-dimensional set.

The redundancy can be visualized as shown in figure 2. The measurement of a single vector in body-fixed axes confines the body's attitude to form a conical solid of revolution about $\mathbf{h}$: those and only those attitudes on the cone would yield the same components b. We shall refer to the set of attitude quaternions consistent with a measurement as the "feasibility cone" Q_b corresponding to that measurement b, *i.e.*, the measurement confines the attitude quaternion $\check{q}$ to lie in Q_b. From the previous discussion, Q_b is one-dimensional and $\check{q}$ has effectively a single degree of freedom. We shall repeatedly draw intuition from the geometry in figure 2 to guide us in the solutions to the stated problems.

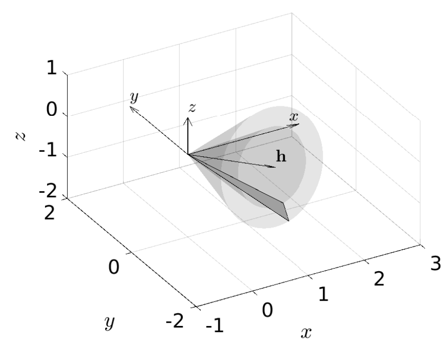

Fig. 2. Possible attitudes of a minimal rigid body formed out of three non collinear points (represented by the triangular patch) consistent with a measurement of a single vector $\mathbf{h}$. The subspace is a cone of revolution about the vector being measured.

2.2 Problem 1. Estimation using measurements of two reference vectors

Let the components of two vectors $\mathbf{h}$ and $\mathbf{k}$ be $a = [a_1\ a_2\ a_3]^T$ and $b = [b_1\ b_2\ b_3]^T$ in the body coordinate system, and $h = [h_1\ h_2\ h_3]^T$ and $k = [k_1\ k_2\ k_3]^T$ in the reference coordinate system respectively. As described above, each reference vector measurement provides two scalar degrees of information regarding the attitude of the rigid body. It is immediately clear that the problem is overconstrained, and we have more equations than unknowns. Geometrically, we have two feasibility cones Q_a and P_b, with the body-axes intersecting along two lines, but with different roll angles for the body about the body-axis. Thus there is no exact solution to this problem in general, unless some of the measurement information is redundant or discarded.

A trivial means to well-pose the problem is to discard components of one of the vector, say $\mathbf{k}$, along the second, $\mathbf{h}$. This is exactly what is done with the TRIAD solution [8], where we use the orthogonal vector triad $\mathbf{h}$, $\mathbf{h} \times \mathbf{k}$, and $\mathbf{h} \times (\mathbf{h} \times \mathbf{k})$ to determine the attitude. A more sophisticated approach is to use all the measurement information – four scalar degrees of information with two reference vector measurements –, and frame the problem as a constrained four-dimensional optimization problem in terms of the quaternion components. This leads to Davenport's q-method and QUEST solutions to Wahba's problem [7].

A novel third approach presented in this paper, is to first determine two solutions $\check{q}$ and $\check{p}$, one each lying on each of the feasibility cones Q_a and P_b corresponding to the measurements a and b, and "closest" to the other cone in some sense. We then fuse the estimates $\check{q}$ and $\check{p}$ appropriately to obtain the final attitude estimate. For example, the final estimate could be obtained using linear spherical interpolation, and the weights be chosen to represent the relative significance attached to the individual measurements.

The first problem can therefore be stated as: *given the measurements a and b in a rotated coordinate system, of the two reference vectors h and k, we would like to estimate the rotated system's two attitude quaternions $\check{q} \in Q_a$ closest (in the least squares sense) to P_b and $\check{p} \in P_b$ closest (in the least squares sense) to Q_a, where Q_a and P_b are the respective feasibility cones.*

2.3 Problem 2. Estimation using rate measurement and measurement of single vector

Suppose we have a measurement of the components $\omega = [\omega_1\ \omega_2\ \omega_3]^T$ of the angular velocity $\boldsymbol{\omega}$ of a moving rigid body, and that we also have a measurement of the components $b = [b_1\ \ b_2\ \ b_3]^T$ of a reference vector $\mathbf{h}$, both measurements being made in the body coordinate system. The components of $\mathbf{h}$ in the reference coordinate system are also known, say $h = [h_1\ h_2\ h_3]^T$. The problem is to make a "best" estimate of the body's attitude $\check{q}$ on the basis of the pair of measurements ω and b, and knowing h.

We shall assume that the initial attitude quaternion is determined using, for *e.g.*, a solution to the first problem or by some other means TRIAD, QUEST, FQA, *etc.* The angular velocity ω can be forward integrated to obtain a "dead-reckoning" estimate of the rotation quaternion. We start with the attitude, $\check{q}(t)$, at time t, and then integrate the differential kinematic equation, to obtain the integrated estimate $\check{p}(t + dt)$. On account of errors in the measurement of ω, this differs from the actual attitude $\check{q}$ of the body. Since we are integrating the errors, the attitude estimates are expected to diverge with time and lead to what is referred to as "drift" in the predicted attitude estimate. Constant errors in the measurement lead to a drift that is proportional to the time of integration, while random white wide-sense stationary noise leads to a drift that is proportional to the square-root of time [26]. Let the error in ω be denoted by the unknown signal $e(t) \in \mathbb{R}^3$ in the body coordinate system. The integrated estimate also has three scalar degrees of error, though it may depend upon e in some complicated path-dependent form.

The second measurement available is b – and of course the knowledge of its reference axes components h. As described at the beginning of this section, this provides two additional scalar degrees of information besides the three from the rate measurement, and constrains the attitude $\check{q}$ to lie in the feasibility cone Q_b. In order to determine the six scalar unknowns, three related to the attitude $\check{q}$, and three related to the integration of the rate measurement error e, we are still lacking one scalar degree of information. In order to specify this degree of freedom and close the problem, we now impose a sixth scalar constraint that uses the attitude $\check{p}$ that was obtained by integrating the kinematic differential equation. We choose that particular $\check{q} \in Q_b$ which is best in the sense that it deviates the least from $\check{p}$.

To summarize, the second problem is *to estimate the attitude quaternion $\check{q}$ which would yield the measurement b in the rotated coordinate system for the reference vector h, and closest (in the least squares sense) to the estimate $\check{p}$ obtained by integrating the angular velocity measurement ω as given in the kinematic differential equation.*

2.4 Nature of measurements of reference vector and angular velocity

The reference vector measurements are assumed to have random, unbiased noise in each of the components, but that they are subsequently normalized for unit magnitude before being passed on to the attitude estimator. This is the most common situation in practice. Any deterministic errors in the measurement are also assumed to be compensated for, *e.g.* acceleration compensation in gravity sense, local field compensation in magnetic field sense.

The angular velocity is not of unit magnitude, in general. Its measurement is also assumed to have random, unbiased noise in each of the components. Deterministic errors in this measurement are also assumed to be compensated for. Compensation of a slowly time-varying gyroscopic bias using the geometric approach has been addressed by the authors in [29].

2.5 Key contributions (for convenience of reviewers)

Having laid the groundwork for both the problems, the detailed solutions follow in the next section. The chief contributions of this paper are contained in Theorem 7 (geometric solution to the first problem), Remarks 7.1, 8.2 (relating the geometric solution to previously reported solutions), Theorem 9 (Solution to the second problem), Remarks 9.2, 10.1 (relating the second geometric solution to previously reported solutions), Theorems 11, and 12 (providing the expressions for the covariance matrix propagation with the geometric method).

3 Attitude quaternion estimation

We first show the equivalence between quaternion displacements and angles, and characterize quaternion orthogonality in terms of rotations, in the following lemmas.

Lemma 1 *The Euclidean distance $\|\check{q} - \check{1}\|$ of an attitude quaternion, $\check{q} = [c_{\Phi/2}\ \ s_{\Phi/2} n]^T$, from the identity element, $\check{1}$, is a positive definite and monotonic function of the magnitude of the principal angle of rotation Φ.*

Proof: This is a simple consequence of the trigonometric half-angle identities.

$$\|\check{q} - \check{1}\|^2 = (c_{\Phi/2} - 1)^2 + s^2_{\Phi/2} = 4\sin^2(\Phi/4),$$

which is a positive definite monotonic function of $\|\Phi\|$ for $\Phi \in [-2\pi, 2\pi]$. A corollary is that the distance $\|\check{q} - \check{p}\| = \|\check{q}^{-1} \otimes \check{p} - \check{1}\|$ between two attitude quaternions is a positive definite and monotonic function of the angle corresponding to the quaternion $\check{q}^{-1} \otimes \check{p}$ that takes $\check{q}$ to $\check{p}$. □

Lemma 2 *Two quaternions are orthogonal if and only if they are related by rotations through π about some axis n.*

$$\check{p}^T \check{q} = 0 \Leftrightarrow \exists n \in \mathbb{R}^3,\ \check{q} = \check{p} \otimes \begin{bmatrix} 0 \\ n \end{bmatrix}. \tag{5}$$

Proof: This follows upon noting that a rotation through π results in the scalar part being zero.

$$\begin{aligned} \check{p}^T \check{q} = 0 &\Leftrightarrow p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3 = 0 \\ &\Leftrightarrow \mathrm{Re}\{\check{q} \otimes \check{p}^{-1}\} = 0 \\ &\Leftrightarrow \exists n \in \mathbb{R}^3,\ \check{q} \otimes \check{p}^{-1} = \begin{bmatrix} 0 \\ n \end{bmatrix}. \qquad \square \end{aligned}$$

We next provide two particular solutions for the simpler problem of estimating the attitude quaternion using a single reference vector measurement, in Lemma 3. We note the algebraic constraint imposed by a vector measurement on the attitude quaternion $\check{q}$. The quaternion $\check{q}$ represents a rigid body rotation, and it transforms the components of the reference vector from h in the reference coordinate system to b in the body-fixed coordinate system:

$$\check{h} = \check{q} \otimes \check{b} \otimes \check{q}^{-1}$$

or

$$\check{q} \otimes \check{b} = \check{h} \otimes \check{q}\,, \tag{6}$$

where the checked quantities $\check{h} = [0\ \ h^T]^T$ and $\check{b} = [0\ b^T]^T$ are the quaternions corresponding to the 3-vectors h and b. Equation (6) expresses the vector measurement constraint as a linear equation in $\check{q}$ subject to a nonlinear normalization constraint.

Lemma 3 *Suppose the components of a reference vector are given by h and b in the reference and body coordinate systems respectively. Let $\Phi = \operatorname{acos} b^T h$, $c = \cos\Phi/2 = \sqrt{(1 + b^T h)/2}$ and $s = \sin\Phi/2 = \sqrt{(1 - b^T h)/2}$. Then, two particular solutions for the body's attitude are given by [19]:*

$$\check{r}_1 = \begin{bmatrix} c \\ s(b \times h)/\|b \times h\| \end{bmatrix},\ \check{r}_2 = \begin{bmatrix} 0 \\ (b + h)/\|b + h\| \end{bmatrix}. \tag{7}$$

Proof: These two solutions are orthogonal in quaternion space, and correspond to the smallest and largest single axis rotations in $[0, \pi]$ that are consistent with the vector measurement in three-dimensional Euclidean space. Geometrically, the first is a rotation through $\operatorname{acos}(b^T h)$ about $(b \times h)/\|b \times h\|$, the second is a rotation through π about $(b + h)/\|b + h\|$. Noting that $\|b \times h\| = \|b\|\|h\| \sin\Phi = \|b\|\|h\| 2sc$, and $\|b + h\| = 2c$, we obtain

$$\begin{aligned} \begin{bmatrix} c \\ (b \times h)/(2c) \end{bmatrix} \otimes \begin{bmatrix} 0 \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ cb + (h - bb^T h)/(2c) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ (b + h)/2c \end{bmatrix} = \begin{bmatrix} 0 \\ h \end{bmatrix} \otimes \begin{bmatrix} c \\ (b \times h)/(2c) \end{bmatrix}, \end{aligned}$$

and

$$\begin{bmatrix} 0 \\ (b + h)/(2c) \end{bmatrix} \otimes \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} -(b^T h)/(2c) \\ (h \times b)/(2c) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ h \end{bmatrix} \otimes \begin{bmatrix} 0 \\ (b+h)/(2c) \end{bmatrix} ,$$

which completes the proof. As a clarification, when $b \to h$, $\check{r}_1$ and $\check{r}_2$ are assumed to take the obvious limits, $\check{1}$ and $\check{h}$, and when $b \to -h$, they are assumed to take the obvious limits, $\check{\imath} = [0\ i]^T$ and $\check{\jmath} = [0\ j]^T$, where $[h\ i\ j]$ is an orthogonal vector triplet. In the latter case ($b + h \to 0$), the orthogonal triad is non-unique, but certain to exist: at least one among the three orthogonal triplets $h, h \times e_x, e_x - h_1 h$; $h, h \times e_y, e_y - h_2 h$; $h, h \times e_z, e_z - h_3 h$ (where $e_x = [1\ 0\ 0]^T, \ldots$) is certain to span $\mathbb{R}^3$, and would be a valid choice for the orthogonal triad $[h\ i\ j]$ after normalization. □

The two special solutions can be rotated by any arbitrary angle about the reference vector h and we would still lie within the feasibility cone, as shown in the next lemma.

Lemma 4 *If $\check{q}$ lies in the feasibility cone Q_b of the measurement b for the reference vector h, then so does any attitude quaternion obtained by rotating $\check{q}$ through an arbitrary angle about h. Conversely, all attitude quaternions lying on the feasibility cone are related to each other by rotations about h.*

Proof: Let Φ be any angle, and let $\check{p}$ be $\check{q}$ rotated through Φ about h, *i.e.*,

$$\check{p} = \begin{bmatrix} c \\ sh \end{bmatrix} \otimes \check{q} ,$$

where $c = \cos \Phi/2$ and $s = \sin \Phi/2$. Then,

$$\check{p} \otimes \check{b} = \begin{bmatrix} c \\ sh \end{bmatrix} \otimes \check{q} \otimes \check{b} = \begin{bmatrix} c \\ sh \end{bmatrix} \otimes \check{h} \otimes \check{q}$$
$$= \check{h} \otimes \begin{bmatrix} c \\ sh \end{bmatrix} \otimes \check{q} = \check{h} \otimes \check{p} .$$

where we have used the fact that two nonzero rotations commute if and only if they are about the same axis. Conversely, $\check{q}^{-1} \otimes \check{h} \otimes \check{q} = b = \check{p}^{-1} \otimes \check{h} \otimes \check{p}$ implies

$$\check{p} \otimes \check{q}^{-1} \otimes \check{h} = \check{h} \otimes \check{p} \otimes \check{q}^{-1}$$
$$\text{or,} \qquad \check{p} \otimes \check{q}^{-1} = \begin{bmatrix} c \\ sh \end{bmatrix} ,$$

for some c and s satisfying $c^2 + s^2 = 1$, which completes the proof. □

Lemma 5 *All elements on the feasibility cone Q_b, of the measurement b for the reference vector h, are in the norm-constrained linear span of the two special solutions in lemma 3.*

Proof: Consider an attitude quaternion $\check{q} = c'\check{r}_1 + s'\check{r}_2$, where $c'^2 + s'^2 = 1$, and $\check{r}_1$ and $\check{r}_2$ are the special solutions of Lemma 3. Then:

$$\begin{bmatrix} cc' \\ \dfrac{c' b \times h + s'(b+h)}{2c} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ b \end{bmatrix}$$
$$= \begin{bmatrix} -s'(1 + 2c^2 - 1)/2c \\ cc'b + \dfrac{c'}{2c}(h - (2c^2 - 1)b) + \dfrac{s'}{2c} h \times b \end{bmatrix}$$
$$= \begin{bmatrix} -cs' \\ \dfrac{c'(h+b) + s' h \times b}{2c} \end{bmatrix}$$
$$= \begin{bmatrix} -s'(2c^2 - 1 + 1)/2c \\ cc'h + \dfrac{c'}{2c}(b - (2c^2 - 1)h) + \dfrac{s'}{2c} h \times b \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ h \end{bmatrix} \otimes \begin{bmatrix} cc' \\ \dfrac{c' b \times h + s'(b+h)}{2c} \end{bmatrix} ,$$

that is, $\check{q} \otimes \check{b} = \check{h} \otimes \check{q}$, which shows that $\check{q}$ is an element on the feasibility cone Q_b. Conversely, any element on the feasibility cone, Q_b, can be written as the composition of $\check{r}$ and a rotation about h through the angle Φ' from lemma 4. Hence,

$$\begin{bmatrix} c' \\ s'h \end{bmatrix} \otimes \begin{bmatrix} c \\ (b \times h)/2c \end{bmatrix}$$
$$= \begin{bmatrix} c'c \\ \dfrac{c'(b \times h)}{2c} + s'ch + \dfrac{s'}{2c}(b - (2c^2 - 1)h) \end{bmatrix}$$
$$= \begin{bmatrix} c'c \\ \dfrac{c'(b \times h) + s'(b+h)}{2c} \end{bmatrix} = c' \begin{bmatrix} c \\ \dfrac{b \times h}{2c} \end{bmatrix} + s' \begin{bmatrix} 0 \\ \dfrac{b+h}{2c} \end{bmatrix} ,$$

which completes the proof. □

It also follows from Lemma 5 that the rotation axis of every rotation on the feasibility cone, Q_b, of the measurement b for the reference vector h, lies on the unit circle containing the vectors $b \times h/\|b \times h\|$, and $(b+h)/\|b+h\|$ (figure 3 left).

Thus, we already see that we have a one dimensional infinity of possible solutions for the attitude quaternion if we have a single reference vector measurement. In fact, the two special solutions provided in lemma 3 are rotations of each other about h through π. In order to obtain a unique solution, we could add either another vector measurement (Wahba's problem), or include an angular velocity measurement (complementary filter).

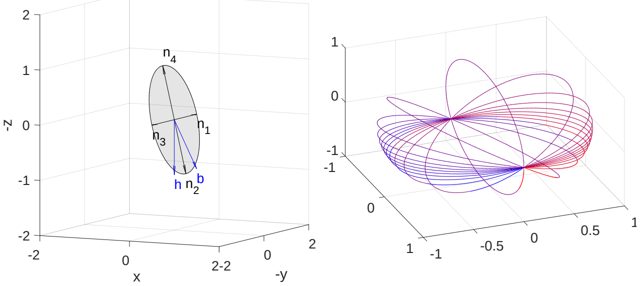

Fig. 3. Left: Possible axes to rotate the rigid body about, in order to measure reference vector h as b in the body axes. The rotation axes lie in the unit great circle spanned by $n_1 = b\times h/\|b\times h\|, n_2 = (b+h)/\|b+h\|, n_3 = -n_1, n_4 = -n_2$. Right: A visual depiction of the covering of the 2-sphere by the body x-axis using all rotations on the feasibility cone, Q_b. The rigid body is being rotated so as to measure the reference vector h as b in the body frame. In order to obtain this measurement, the body may be rotated (by differing amounts) about the set of unit vectors spanned by n_1 and n_2.

We note one final result about the feasibility cone subspace.

Lemma 6 *Any two unequal attitude quaternions, $\check{p}$ and $\check{q}$, define the feasibility cone corresponding to some vector meaurement.*

Proof: The claim follows trivially upon noting that rotations about the same axis commute, and the axis n of $\check{q}\otimes\check{p}^{-1}$ is the reference direction whose body frame measurements are the same with both $\check{p}$ and $\check{q}$:

$$\check{q}\otimes\check{p}^{-1} = \begin{bmatrix} c \\ sn \end{bmatrix},$$
$$\Rightarrow \check{q}\otimes\check{p}^{-1}\otimes\check{n} = \check{n}\otimes\check{q}\otimes\check{p}^{-1}$$
$$\Rightarrow \check{p}^{-1}\otimes\check{n}\otimes\check{p} = \check{q}^{-1}\otimes\check{n}\otimes\check{q}. \qquad \square$$

3.1 Attitude estimation using two vector measurements

We now derive a unique solution for the attitude quaternion when we have measurements of two reference vectors and would like to incorporate both of them in deriving the attitude estimate. Let a and b be the body-referred components of reference vectors h and k ($h, k \in \mathbb{S}^2$ contain the components of the two vectors along some reference coordinate axes) respectively. Suppose the rotation quaternion is estimated to be $\check{q} = [q_0 \;\; q]^T$ on the basis of a, and it is independently estimated to be $\check{p} = [p_0 \;\; p]^T$ on the basis of b, both estimates being obtained by applying, say, Lemma 3.

The estimates $\check{q}$ and $\check{p}$ are each indeterminate to one scalar degree of freedom as shown in lemma 4: a rotation about the corresponding vectors h and k respectively. Let these rotations be given by the quaternions $\check{r}_1 = [c_1 \;\; s_1 h]^T$ and $\check{r}_2 = [c_2 \;\; s_2 k]^T$ respectively where $c_i = \cos\Phi_i/2$ and $s_i = \sin\Phi_i/2$ for $i \in \{1, 2\}$. The problem is to determine the optimal values of Φ_1 and Φ_2 so as to minimize the displacement from the rotated $\check{r}_1\otimes\check{q}$ to $\check{r}_2\otimes\check{p}$.

$$\check{r}_1\otimes\check{q} = \begin{bmatrix} c_1 \\ s_1 h \end{bmatrix}\otimes\begin{bmatrix} q_0 \\ q \end{bmatrix} = \begin{bmatrix} c_1 q_0 - s_1 q^T h \\ c_1 q + s_1 q_0 h + s_1 h\times q \end{bmatrix},$$
$$\check{r}_2\otimes\check{p} = \begin{bmatrix} c_2 \\ s_2 k \end{bmatrix}\otimes\begin{bmatrix} p_0 \\ p \end{bmatrix} = \begin{bmatrix} c_2 p_0 - s_2 p^T k \\ c_2 p + s_2 p_0 k + s_2 k\times p \end{bmatrix}. \quad (8)$$

We could either minimize $\|\check{r}_1\otimes\check{q} - \check{r}_2\otimes\check{p}\|^2$, or equivalently from Lemma 1, maximize the first component of $(\check{r}_1\otimes\check{q})^{-1}\otimes\check{r}_2\otimes\check{p}$. In order to keep the reasoning straightforward, we choose the former. So we need to minimize the cost function

$$\begin{aligned} J(\Phi_1,\Phi_2) &= (c_1 q_0 - s_1 q^T h - c_2 p_0 + s_2 p^T k)^2 \\ &\quad + \|c_1 q + s_1(q_0 h + h\times q) - c_2 p - s_2(p_0 k + k\times p)\|^2, \\ &= 2 + 2l_1 c_1 c_2 + 2l_2 s_1 s_2 + 2l_3 c_1 s_2 + 2l_4 s_1 c_2, \qquad (9) \end{aligned}$$

where $l_1 = -q_0 p_0 - q^T p$, $l_2 = (-q_0 p^T + p_0 q^T - (q\times p)^T)h\times k - (q_0 p_0 + q^T p)h^T k$, $l_3 = k^T(q_0 p - p_0 q + q\times p)$, and $l_4 = h^T(p_0 q - q_0 p + p\times q)$, are known quantities. Now minimizing the cost function with respect to the independent pair of variables $\Phi_1 + \Phi_2$ and $\Phi_1 - \Phi_2$ yields

$$\begin{bmatrix} \Phi_1 - \Phi_2 \\ \Phi_1 + \Phi_2 \end{bmatrix} = 2\begin{bmatrix} \mathrm{atan2}(l_3 - l_4, -(l_1 + l_2)) \\ \mathrm{atan2}(-(l_3 + l_4), l_2 - l_1) \end{bmatrix}. \qquad (10)$$

Equation (10) can be solved for Φ_1, and Φ_2, and that completes the solution. The above derivation can be summarized in the form of the following theorem:

Theorem 7 *If $\check{q}$ and $\check{p}$ are any two special attitude estimates for a rotated system, derived independently using the measurements a and b in the body-fixed coordinate system of two linearly independent reference vectors h and k respectively, then the optimal estimate incorporating the measurement b in $\check{q}$ is $\check{r}_1\otimes\check{q}$, and the optimal estimate incorporating the measurement a in $\check{p}$ is given by $\check{r}_2\otimes\check{p}$, where $\check{r}_1 = [c_1 \;\; s_1 h]^T$ and $\check{r}_2 = [c_2 \;\; s_2 k]^T$, $c_i = \cos\Phi_i$, $s_i = \sin\Phi_i$, and Φ_1 and Φ_2 are given by equation (10).*

Proof: The proof follows from the construction leading to equations (8, 10). Refer figure 4. $\square$

Remark 7.1 *Relation to the TRIAD attitude estimate [8]*: The attitude estimates $\check{r}_1\otimes\check{q}$ and $\check{r}_2\otimes\check{p}$, where $\check{r}_1 = [c_1 \;\; s_1 h]^T$ and $\check{r}_2 = [c_2 \;\; s_2 k]^T$, are the same as the TRIAD solution in literature [15]. Each of them individually yields an estimate that is competely consistent with one measurement, but only partially consistent with the other.

Corollary 8 *The rotation from the TRIAD estimate* $\check{r}_1 \otimes \check{q}$ *to* $\check{r}_2 \otimes \check{p}$ *in (10) is about an axis perpendicular to both* h *and* k*.*

Proof: Let $\check{q}' = \check{r}_1 \otimes \check{q}$ and $\check{p}' = \check{r}_2 \otimes \check{p}$ be the optimal TRIAD estimates. Let us now optimize upon these optimal estimates. That should return no required corrections, *i.e.* $\check{r}'_1 = \check{r}'_2 = \check{1}$. This is equivalent to saying $\Phi'_1 = \Phi'_2 = 0$. This in turn is equivalent to $l'_3 = l'_4 = 0$, or $h^T(p'_0 q' - q'_0 p' + p' \times q') = k^T(q'_0 p' - p'_0 q' + q' \times p') = 0$. But then $q'_0 p' - p'_0 q' - p' \times q'$ is just the vector portion of $\check{p}' \otimes \check{q}'^{-1}$, the rotation taking the optimal TRIAD estimate $\check{q}'$ to $\check{p}'$ in the reference coordinate system. □

Remark 8.1 *Geometric filtering between the TRIAD estimates*: In order to filter the noise in the vector measurements, we could now interpolate between the two solutions obtained in equations (8, 10). Let $\check{q}, \check{p}$ be the TRIAD attitude estimates (denoted as $\check{q}'$ and $\check{p}'$ in Corollary 8) using vector measurements a and b of h and k respectively, and $x \in [0, 1] \subset \mathbb{R}$. The interpolated quaternion, $\check{q}_f$, from $\check{q}$ to $\check{p}$ is given by any of the following four equivalent expressions [4]:

$$\begin{aligned} \check{q}_f &= \check{q} \otimes (\check{q}^{-1} \otimes \check{p})^x = \check{p} \otimes (\check{p}^{-1} \otimes \check{q})^{1-x} \\ &= (\check{q} \otimes \check{p}^{-1})^{1-x} \otimes \check{p} = (\check{p} \otimes \check{q}^{-1})^x \otimes \check{q}\,. \end{aligned} \tag{11}$$

The interpolation ratio x is now choosen to perform a desired weighting of the two TRIAD estimates $\check{q}$ and $\check{p}$ in the final result. When the noise in each of the measurements a and b is zero-mean Gaussian with variance σ_i^2, the appropriate choice for x would be $\sigma_a^2/(\sigma_a^2 + \sigma_b^2)$.

Remark 8.2 *Relation to the solutions of Wahba's problem [11]*: Let the TRIAD estimates again be denoted as $\check{q}$ and $\check{p}$. Further let $\check{r} = \check{p} \otimes \check{q}^{-1}$ denote the rotation that takes $\check{q}$ to $\check{p}$ in the reference coordinate system. From Corollary 8, we know that $\check{r} = [c_{\Phi/2}\ s_{\Phi/2}(h \times k)^T/\|h \times k\|]^T$ for some Φ. Next, let $\check{w}$ be the solution to Wahba's problem, that minimizes the loss function $\alpha\|\check{w} \otimes \check{a} \otimes \check{w}^{-1} - \check{h}\|^2 + \beta\|\check{w} \otimes \check{b} \otimes \check{w}^{-1} - \check{k}\|^2$. Now $\check{w}$ must lie on the feasibility cone containing $\check{q}$ and $\check{p}$. Otherwise, we could move it towards the cone so as to reduce both the errors $\|\check{w} \otimes \check{a} \otimes \check{w}^{-1} - \check{h}\|^2$ and $\|\check{w} \otimes \check{b} \otimes \check{w}^{-1} - \check{k}\|^2$ in the loss function. So, if $\check{w} \otimes \check{q}^{-1}$ and $\check{p} \otimes \check{w}^{-1}$ rotate the body through Φ_q and Φ_p about $h \times k$, then we must have $\Phi_q + \Phi_p = \Phi$. The loss function would be $2\alpha(1 - \cos\Phi_q) + 2\beta(1 - \cos\Phi_p)$. Thus the solution to Wahba's problem maximizes $\alpha\cos\Phi_q + \beta\cos\Phi_p$, subject to $\Phi_p + \Phi_q = \Phi$: $-\alpha\sin\Phi_q + \beta\sin(\Phi - \Phi_q) = 0 \Rightarrow \tan\Phi_q = \sin\Phi/(\alpha/\beta + \cos\Phi)$ and $\tan\Phi_p = \sin\Phi/(\beta/\alpha + \cos\Phi)$. The filtered estimate $\check{q}_f$ may be derived as the rotation through Φ_q about $h \times k$ from $\check{q}$, or $-\Phi_p$ about $h \times k$ from $\check{p}$.

Remark 8.3 *Incorporating hard inequality constraints*: Since the presented solution is geometric in nature, it is straightforward to include geometric constraints on the solution. For instance, some attitude estimation problems have hard constraints [14], [25]. In control solutions, such constraints are most often enforced using Barrier Lyapunov functions (BLFs) [13] for bounded solutions. Such a strategy can easily be employed in our framework, in contrast with the linear algebraic solutions which are more suitable to handle quadratic forms. Instead of determining the interpolaton factor x using the noise variance, it can be determined as the argument that minimizes a cost function that contains a BLF:

$$x = \operatorname*{argmin}_{x \in [0,1]}(\alpha \sec(x/a) + (1 - x)^2), \tag{12}$$

where α and a are appropriately chosen constants. It may be appreciated that the cost function can be any infinite potential well, and not just the above formulation. This generality is enabled by the simple interpolation of the geometric angle between the two solutions of theorem 7.

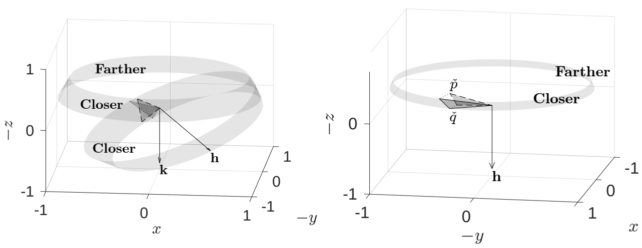

Fig. 4. A visual depiction of the solutions presented in Theorems 7 and 9. The image on the left shows the two solutions $\check{r}_1 \otimes \check{q}$ (dotted triangle) and $\check{r}_2 \otimes \check{p}$ (dashed triangle) of theorem 3. The figure on the right shows the solution $\check{q}$ (solid triangle) of theorem 4 obtained by projecting the integrated attitude $\check{p}$ (dashed triangle) onto the feasibility cone of vector measurement b.

3.2 *Attitude estimation using single vector measurement and rate measurement*

We first write down the constraints imposed by the measurement upon the attitude quaternion $\check{q} = [c\ s[\mathbf{n}]]^T = [c\ sn_1\ sn_2\ sn_2]^T$, where $c = \cos(\Phi/2)$ and $s = \sin(\Phi/2)$ are functions of the rotation angle Φ, and $\mathbf{n}$ is a unit vector along the rotation axis with components $n = [n_1\ n_2\ n_3]^T$ in the reference coordinate system. The constraint is given in equation (6). Converting the quaternion multiplication to vector notation, equation (6) can also be written as:

$$\begin{bmatrix} -sn^T b \\ cb + s[n\times]b \end{bmatrix} = \begin{bmatrix} -sh^T n \\ ch + s[h\times]n \end{bmatrix},$$

i.e.,
$$\begin{bmatrix} -s(h - b)^T n \\ c(h - b) + s[(h + b)\times]n \end{bmatrix} = 0\,,$$

where $[n\times]$ denotes the cross product matrix associated with the 3-vector n. Expanding the vectors,

$$\begin{bmatrix} & -f_1 & -f_2 & -f_3 \\ f_1 & & -g_3 & g_2 \\ f_2 & g_3 & & -g_1 \\ f_3 & -g_2 & g_1 & \end{bmatrix} \begin{bmatrix} c \\ sn_1 \\ sn_2 \\ sn_3 \end{bmatrix} = 0 \ , \tag{13}$$

where $\quad f = h - b \quad$ and $\quad g = h + b \ ,$

so that $\quad f_1 g_1 + f_2 g_2 + f_3 g_3 = f^T g = h^T h - b^T b = 0 \ .$

While it is not obvious, equation (13) has a double redundancy, so the system of four linear equations actually has rank 2 and nullity 2. This can be seen by considering the solution:

$$\check{q} = \begin{bmatrix} c \\ (-cf_2 + sn_3 g_1)/g_3 \\ (cf_1 + sn_3 g_2)/g_3 \\ sn_3 \end{bmatrix} \ , \tag{14}$$

where sn_1 and sn_2 are solved in terms of c and sn_3 using the inner two row equations in equation (13). Substituting these in the outer two rows of equation (13) satisfies them trivially, so these two rows do not yield any additional information. This makes sense as we have not yet imposed the normalization constraint that $n_1^2 + n_2^2 + n_3^2 = 1$ (c and s, representing $\cos\Phi/2$ and $\sin\Phi/2$, are already assumed to satisfy $c^2 + s^2 = 1$). And we are anyway to end up with one degree of freedom in $\check{q}$ if using the vector measurement constraint alone, as discussed earlier.

We now move on to utilizing the angular velocity measurement that determines the differential evolution of the attitude. The kinematic differential equation for the quaternion is the linear first order ODE:

$$\dot{\check{q}} = \frac{1}{2}\check{q} \otimes \check{\omega} = \frac{W\check{q}}{2}, \tag{15}$$

where $\check{\omega}$ is the quaternion form of the 3-vector ω. In continuous time, the integration of (15) for a constant W gives an estimate $\check{p}(t+T) = \exp(WT/2)\check{q}(t)$. For example, if $\omega(t+s) = [0 \ \ (\xi\cos\xi s) \ \ 0]^T$, then $\check{p} = \exp\{j\sin\xi T/2\}\check{q} = \cos(\sin(\xi T)/2)\check{q} + j\sin(\sin(\xi T)/2)\check{q}$, where $j = [\otimes \check{e}_2]$, where $\check{e}_2 = [0 \ 0 \ 1 \ 0]^T$. For a time-varying ω, the state transition matrix replaces the exponential. In discrete time, denoting the integrated estimate as $\check{p}(i+1)$, the above equation takes the form

$$\check{p}(i+1) = \check{q}(i) + \frac{T}{2}\check{q}(i) \otimes \check{\omega}(i) \ , \tag{16}$$

where T is the time step from the previous estimation of $\check{q}(i)$ to the current estimation $\check{p}(i+1)$. In the subsequent derivation, we shall omit the time argument of $\check{p}$, as there is no ambiguity.

The deviation of the vector-aligned quaternion estimate, $\check{q}$ in equation (14), from the integrated estimate, $\check{p}$ in equation (16), can be expressed as the difference of $\check{p}^{-1} \otimes \check{q}$ from $\check{1}$. But minimizing the distance of a quaternion from the unit quaternion is the same as minimizing the rotation angle Φ (Lemma 1), which is, in turn, the same as maximizing the zeroeth component of the quaternion, $\cos(\Phi/2)$. Note that, the quaternions $\check{p}^{-1} \otimes \check{q}$ and $-\check{p}^{-1} \otimes \check{q}$ affect the same rigid body rotation in 3-dimensional Euclidean space, but minimizing the distance of one from $\check{1}$ maximizes the distance of the other in quaternion space. So we just extremize the distance, rather than specifically minimize it. Once we have the solution set, we can check which solutions correspond to a maximum and which to a minimum, and choose the latter.

We therefore need to extremize the zeroeth component of $\check{p}^{-1}\check{q}$, where $\check{p} = [p_0 \ p_1 \ p_2 \ p_3]^T$ is the attitude estimate obtained by integrating the angular velocity ω as given in equation (15) and $\check{q}$ is expressed in terms of c/s and n_3 as in equation (14), while enforcing the constraint in equation (6). This can be accomplished by using the method of Lagrange multipliers to define a cost function that invokes the error norm as well as the constraint. Below, we have multiplied the cost function by the constant g_3 and the constraint by g_3^2, noting that the solution is unaffected by such a scaling:

$$\begin{aligned} J(\Phi, n_3) &= g_3[\check{p}^{-1} \otimes \check{q}]_0 + \lambda g_3^2(n_1^2 + n_2^2 + n_3^2 - 1) \\ &= (cp_0 + sn_3 p_3)g_3 + (-cf_2 + sn_3 g_1)p_1 + (cf_1 + sn_3 g_2)p_2 \\ &\qquad + \lambda\left(n_3^2 g^T g + 2\frac{cn_3}{s}(f_1 g_2 - f_2 g_1) + \frac{c^2}{s^2}(f_1^2 + f_2^2) - g_3^2\right) \\ &= c(g_3 p_0 + f_1 p_2 - f_2 p_1) + sn_3 g^T p \\ &\quad + \lambda\left(n_3^2 g^T g + 2\frac{cn_3}{s}(f_1 g_2 - f_2 g_1) + \frac{c^2}{s^2}(f_1^2 + f_2^2) - g_3^2\right) , \end{aligned}$$

where p denotes the vector portion of $\check{p}$. Now we set the first order partial derivatives of J to 0:

$$\begin{aligned} 0 = \partial_\Phi J &= -s(g_3 p_0 + f_1 p_2 - f_2 p_1) + cn_3 g^T p \\ &\quad + \left(-\frac{2\lambda}{s^2}\right)\left(\frac{c}{s}(f_1^2 + f_2^2) + n_3(f_1 g_2 - f_2 g_1)\right) , \end{aligned} \tag{17}$$

$$0 = \partial_{n3} J = sg^T p + 2\lambda g^T g n_3 + 2\lambda\frac{c}{s}(f_1 g_2 - f_2 g_1), \tag{18}$$

$$0 = \partial_\lambda J = n_3^2 g^T g - g_3^2 + \frac{2cn_3}{s}(f_1 g_2 - f_2 g_1) + \frac{c^2}{s^2}(f_1^2 + f_1^2). \tag{19}$$

This yields, for the ratio $\kappa = c/sn_3$:

$$\kappa = \frac{p_0 g^T g + p^T g \times f}{p_0(g_1 f_2 - g_2 f_1) + \sum_{1,2} p_i(g_i g_3 - f_i f_3) + p_3(f_1^2 + f_2^2 + g_3^2)}. \tag{20}$$

Fortuituously, $c/s = \cot(\Phi/2)$ is therefore just proportional to n_3, and upon expressing c/s in terms of n_3 in the normalization constraint (equation (19)), the resulting equation becomes extremely simple to solve:

$$g_3^2 = g^T g n_3^2 + 2\kappa(f_1 g_2 - f_2 g_1) n_3^2 + \kappa^2 n_3^2 (f_1^2 + f_2^2) \ ,$$

or

$$n_3 = \frac{g_3}{\sqrt{g^T g + 2\kappa(f_1 g_2 - f_2 g_1) + \kappa^2 (f_1^2 + f_2^2)}} \ , \quad (21)$$

$$\frac{c}{s} = \frac{\kappa g_3}{\sqrt{g^T g + 2\kappa(f_1 g_2 - f_2 g_1) + \kappa^2 (f_1^2 + f_2^2)}} \ . \quad (22)$$

The other components of the attitude quaternion can be obtained using the inner two rows of equation (14). Thus we obtain the following theorem.

Theorem 9 *If the angular velocity of a rigid body is integrated to yield a attitude quaternion estimate $\breve{p}$, then the estimate $\breve{q} \in Q_b$ lying in the feasibility cone of measurement b which is closest to $\breve{p}$, is given by equations (14, 20, 21, 22).*

Proof: The proof follows from the construction leading to equations (14, 20, 21, 22). Refer figure 4. □

Remark 9.1 *Solution when reference vector is aligned with z-axis*: A common application of the presented solution would be to an aerial robot that uses an accelerometer to measure the gravity vector (after acceleration compensation). Since the reference coordinate system's z-axis is aligned with the reference vector $\mathbf{h}$, we have $f = [(-b_1)\ (-b_2)\ (1-b_3)]^T$ and $g = [b_1\ b_2\ (1+b_3)]^T$. Equations (20, 22) now simplify to:

$$\kappa = \frac{c}{s n_3} = \frac{(1+b_3)p_0 - b_1 p_2 + b_2 p_1}{b_1 p_1 + b_2 p_2 + (1+b_3) p_3} \ ,$$

$$\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} c \\ s n_1 \\ s n_2 \\ s n_3 \end{bmatrix} = \frac{1}{\sqrt{2(1+\kappa^2)(1+b_3)}} \begin{bmatrix} \kappa(1+b_3) \\ \kappa b_2 + b_1 \\ -\kappa b_1 + b_2 \\ (1+b_3) \end{bmatrix} , \quad (23)$$

where we have used the fact that $(1+b_3)^2 + b_1^2 + b_2^2 = 2(1+b_3)$. While the introduction of the auxillary variable κ in equations (20 - 22) seems adhoc, its role is more clearly visible now – κ parameterizes the feasibility cone Q_b in terms of the two special solutions provided in lemma 3:

$$\breve{q} = \frac{\kappa \breve{r}_1 + \breve{r}_2}{\sqrt{1+\kappa^2}} = \frac{(\breve{r}_1 \breve{r}_1^T + \breve{r}_2 \breve{r}_2^T)\breve{p}}{\|(\breve{r}_1 \breve{r}_1^T + \breve{r}_2 \breve{r}_2^T)\breve{p}\|} \ . \quad (24)$$

Remark 9.2 *Relation to the EKF [5]*: A filtered attitude estimate $\breve{q}_f$ can be obtained by projecting the integrated estimate, $\breve{p}$, onto the feasibility cone corresponding to a filtered vector measurement b_f, to yield the vector aligned estimate $\breve{q}$ of Theorem 9. The predict-step in Theorem 9 is identical to that in the EKF: we just integrate the dynamics of the state from the previous time step. Note that the EKF accommodates nonlinearity in the dynamics in the prediction step, and so it is okay for the attitude dynamics to be bilinear in the state (attitude) and input (angular velocity). It is the correction step where the geometric method diverges from the EKF. It may be noted that the projection onto the feasibility cone affects only two degrees of freedom of the attitude. The attitude degree of freedom associated with rotation about the reference vector is completely unaffected by the projection. Thus the filtering may be precisely accomplished by implementing it upon the vector measurement. A detailed derivation of the filtered vector measurement and the propagation of the covariance matrices is given in the next subsection, and the improvement in performance is demonstrated using simulations in section 5.

The following corollary follows from theorem 9.

Corollary 10 *The correction that takes the integrated estimate $\breve{p}$ into the feasibility cone Q_b is essentially a rotation about an axis that is orthogonal to the reference vector h.*

Proof: With the simplifying choice for the reference coordinate system's z-axis that leads to equation (23), the proof is simple. The correcting rotation in the reference coordinate system is:

$$\breve{r} = \breve{q} \otimes \breve{p}^{-1} = \begin{bmatrix} \kappa(1+b_3) \\ \kappa b_2 + b_1 \\ -\kappa b_1 + b_2 \\ (1+b_3) \end{bmatrix} \otimes \begin{bmatrix} p_0 \\ -p_1 \\ -p_2 \\ -p_3 \end{bmatrix} / \sqrt{2(1+\kappa^2)(1+b_3)} \, .$$

So, using the expression for κ in equation (23), we obtain $r_3 = 0$. In the general case of an arbitrary h, the proof is more tedious, but still valid [28]. The underlying reason for this result is just that a rotation about any other axis would have an unnecessary component about h, and that would make the correction to reach Q_b suboptimal. □

An elegant expression for the corrected attitude estimate $\breve{q}$ in terms of the integrated estimate $\breve{p}$ and vector measurement b of a single vector is:

$$\breve{q} = \frac{\breve{p} - \breve{h} \otimes \breve{p} \otimes \breve{b}}{\|\breve{p} - \breve{h} \otimes \breve{p} \otimes \breve{b}\|} . \quad (25)$$

Equation (25) is directly consistent with the measurement constraint $\breve{h} \otimes \breve{q} = (\breve{h} \otimes \breve{p} + \breve{p} \otimes \breve{b}) / \|\breve{h} \otimes \breve{p} + \breve{p} \otimes \breve{b}\| = \breve{q} \otimes \breve{b}$, so it lies on the feasibility cone by definition. At the same time, the correction $\breve{q} \otimes \breve{p}^{-1}$ in the reference coordinate system is about an axis perpendicular to h as required by Corollary 10.

Equation (25) may rigorously be derived from equation (24) as follows:

$$2(\check{r}_1\check{r}_1^T + \check{r}_2\check{r}_2^T) = 1_{4\times 4} - [h\otimes][\otimes\check{b}],$$
$$\Rightarrow (\check{r}_1\check{r}_1^T + \check{r}_2\check{r}_2^T)\check{p} = (\check{p} - \check{h}\otimes\check{p}\otimes\check{b})/2,$$

which upon normalizing yields the stated equivalence between (25) and (24). Note that

$$\begin{aligned} 1 - [h\otimes][\otimes\check{b}] &= 1 - \begin{bmatrix} & -h^T \\ h & [h\times] \end{bmatrix} \begin{bmatrix} & -b^T \\ b & -[b\times] \end{bmatrix} \\ &= \begin{bmatrix} 1 + h^T b & (b\times h)^T \\ b\times h & (1 - h^T b) + hb^T + bh^T \end{bmatrix}. \end{aligned}$$

Remark 10.1 *Relation to the Explicit complementary filter (ECF) [21]*: The ECF in [21] Theorem 5.1 may be realized out of Theorem 9 by noting that the correction quaternion in the body frame is given by:

$$\check{p}^{-1}\otimes\check{q} = \frac{\check{1} - \check{p}^{-1}\otimes\check{h}\otimes\check{p}\otimes\check{b}}{\|\check{p} - \check{h}\otimes\check{p}\otimes\check{b}\|} = \frac{\check{1} - \check{b}_p\otimes\check{b}}{\|\check{p} - \check{h}\otimes\check{p}\otimes\check{b}\|}, \quad (26)$$

where, $\check{b}_p = \check{p}^{-1}\otimes\check{h}\otimes\check{p}$ is the expected measurement of $\mathbf{h}$ in the body frame, if $\check{p}$ was already the correct attitude. On the other hand, the correction from the integrated estimate can be obtained by including a correction term ω_c in the angular velocity such that:

$$\frac{\check{q} - \check{p}}{T} = \frac{1}{2}\check{p}\otimes\breve{\omega}_c,$$

where $\breve{\omega}_c$ is the equivalent correction required in the angular velocity over a time-step T. Let $b_p^T b = 2c^2 - 1$. For small corrections, $b_p \approx b$, and so the incremental correction angular velocity is given to first order by:

$$\begin{aligned} \breve{\omega}_c &= \frac{2}{T}\left[\check{p}^{-1}\otimes\check{q} - \check{1}\right] = \frac{2}{T}\left[\frac{\check{1} - \check{b}_p\otimes\check{b}}{\|\check{p} - \check{h}\otimes\check{p}\otimes\check{b}\|} - \check{1}\right], \\ &= \frac{2}{T}\left(\begin{bmatrix} c \\ b\times b_p/2c \end{bmatrix} - \check{1}\right) \approx \frac{1}{T}\begin{bmatrix} 0 \\ b\times b_p \end{bmatrix}, \end{aligned}$$

whose vector portion is exactly the same as that reported in [21] Theorem 5.1, with the gain k_P equal to the time step $1/T$. Note that this also ensures that [21] Theorem 5.1 is dimensionally consistent: k_P must have dimensions of reciprocal time. For values of k_P larger than $1/T$, we obtain a larger correction $\breve{\omega}_c$, and a larger weight for measurement b in the final filtered estimate.

4 Effect of noise in measurements on the estimation

Kalman filtering during the update-step is typically implemented on the state estimate, as this is the most intuitive interpretation in linear systems. The translation to the space of attitude quaternions is straightforward, but inefficient. Since the vector measurement b confines the state to the coresponding feasibility cone Q_b, the interpolation between the integrated estimate $\check{p}$ and the corrected estimate $\check{q}$ is computationally expensive upon accounting for singularity of the covariance matrix. An equivalent and more elegant alternative is to consider interpolating between the predicted vector measurement $\check{b}_p = \check{p}^{-1}\otimes\check{h}\otimes\check{p}$ and the actual measurement b. The two approaches are shown in figure 5.

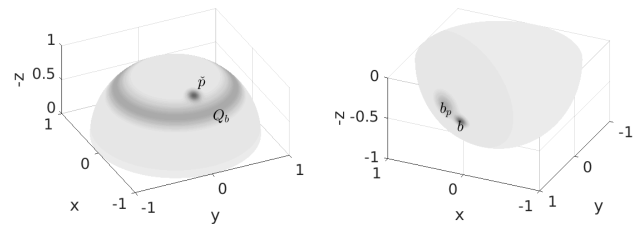

Fig. 5. Left: Traditional Kalman filtering translated to attitude estimation involves interpolating between the integrated attitude estimate $\check{p}$ and a feasibility cone Q_b of attitudes corresponding to a vector measurement. The 3-sphere attitude space has been projected on a 2-sphere for visualization purposes (by, for example, ignoring the roll component). Right: A computationally efficient alternative is to filter the vector measurement b with respect to its predicted value b_p.

Let us first describe a filter on the vector measurement b using the angular velocity information. Suppose the angular velocity is integrated to yield the attitude estimate $\check{p} = [p_0\ p^T]^T$. This attitude then predicts the body-frame components b_p for the reference vector h through equation (6). Perturbations in $\check{p}$ induce perturbations in the predicted vector measurement b_p:

$$\begin{aligned} b_p &= \left[(p_0^2 - p^T p)1_{3\times 3} + 2pp^T - 2p_0[p\times]\right] h, \\ \Rightarrow \frac{\delta b_p}{\delta\check{p}} &= 2\left[(p_0 h + h\times p)\ \ (p^T h + ph^T - hp^T + p_0[h\times])\right] \\ &= \nabla_p b_p. \end{aligned} \quad (27)$$

The above equation yields the covariance B_p of the predicted vector measurement in terms of the covariance Π of the predicted attitude estimate.

$$B_p = \nabla_p b_p \Pi \nabla_p^T b_p. \quad (28)$$

An expression for the covariance matrix Π of the integrated estimate $\check{p}$ may be obtained from the kinematic equation (16) for small time-steps.

$$\Pi = \Xi + \frac{T^2}{4}\begin{bmatrix} q_0 & -q^T \\ q & q_0 + [q\times] \end{bmatrix} W \begin{bmatrix} q_0 & q^T \\ -q & q_0 - [q\times] \end{bmatrix}, \quad (29)$$

where Ξ and W are the covariances of the attitude estimate at the previous time step and the angular velocity measurement.

The filtered vector measurement is given by fusing the two estimates:

$$b_f = (B + B_p)^{-1}(Bb_p + B_p b), \qquad (30)$$

where B and B_p are covariance matrices corresponding to the actual vector measurement b and the predicted vector measurement b_p. The covariance matrix of the fused measurement is:

$$B_f = (B + B_p)^{-1}(BB_pB + B_pBB_p)(B + B_p)^{-1}. \qquad (31)$$

The filtered and corrected attitude estimate is then obtained by projecting the integrated estimate $\check{p}$ onto the feasibility cone Q_{bf} corresponding to the filtered vector measurement b_f.

We now analyze the effect of independent, unbiased noise in the angular velocity measurement ω and vector measurement b on the estimated attitude $\check{q}$, in order to determine expressions for the propagation of the covariance matrix. In particular, we shall assume that there is no bias error in ω. Further, we shall make the reasonable assumption that the errors are small enough relative to the norms of the quantities to consider them as perturbations, and therefore add the effects of individual noise sources to obtain the cumulative effect. The analysis in this section enables the derivation of expressions yielding the propagation of the covariance matrices upon filtering the attitude estimate using sequential angular velocity and vector measurements.

We shall introduce some new notation, to avoid lengthy expressions. The quaternion attitude estimate is given by equation (23):

$$\sqrt{2(\alpha^2+\beta^2)(1+b_3)}\,\check{q} = \begin{bmatrix} \alpha(1+b_3) \\ \alpha b_2 + \beta b_1 \\ -\alpha b_1 + \beta b_2 \\ \beta(1+b_3) \end{bmatrix} = \alpha\check{u} + \beta\check{v}\,, \qquad (32)$$

where $\check{u} = [(1+b_3)\ b_2\ -b_1\ 0]^T$ and $\check{v} = [0\ b_1\ b_2\ (1+b_3)]^T$ are scaled versions of the two special solutions from Lemma 3, $\alpha = p_0(1+b_3) + p_1b_2 - p_2b_1 = \check{p}^T\check{u}$, and $\beta = p_1b_1 + p_2b_2 + p_3(1+b_3) = \check{p}^T\check{v}$.

Let us first consider the effect of noise in ω alone. Suppose the noise in ω leads to a small error in the integrated estimate $\delta\check{p} = (T/2)\check{q}\otimes\delta\check{\omega}$ (refer equation (16) for a small T). The errors in α, β would then be:

$$\begin{bmatrix}\delta\alpha \\ \delta\beta\end{bmatrix} = \begin{bmatrix} 1+b_3 & b_2 & -b_1 & \\ & b_1 & b_2 & 1+b_3 \end{bmatrix}\delta\check{p} = \begin{bmatrix}\check{u}^T \\ \check{v}^T\end{bmatrix}\delta\check{p}\,.$$

Theorem 11 *In the absence of any other errors, a perturbation error $\delta\check{p}$ in the integrated attitude estimate $\check{p}$ leads to a perturbation in the vector-aligned attitude estimate $\check{q}$ (equation (23)) equal to the projection of $\delta\check{p}$ onto the feasibility cone, i.e., the subspace spanned by the two special solutions in lemma 3, and orthogonal to the nominal attitude estimate.*

Proof: Taking differentials of equation (32):

$$\sqrt{2(\alpha^2+\beta^2)(1+b_3)}\,\delta\check{q} = \left\{\begin{array}{r} -\dfrac{(\alpha\delta\alpha+\beta\delta\beta)\sqrt{2(1+b_3)}}{\sqrt{\alpha^2+\beta^2}}\check{q} \\ +\check{u}\delta\alpha + \check{v}\delta\beta \end{array}\right.,$$

$$= (1 - \sqrt{\frac{2(1+b_3)}{\alpha^2+\beta^2}}\check{q}\check{p}^T)(\check{u}\delta\alpha + \check{v}\delta\beta)\,. \qquad (33)$$

Once we have expressed the error as the sum of first order differentials, the multiplying coefficients may now be approximated to their nominal values – any error on account of the approximation would be multiplied by the differentials and therefore be of higher order. Specifically, we may approximate $\check{p}\approx\check{q}$, so $\check{p}\otimes\check{b}\approx\check{h}\otimes\check{p}$, so $\alpha\approx 2p_0\approx 2q_0$, and $\beta\approx 2p_3\approx 2q_3$, in the coefficients, to obtain

$$\delta\check{q} = (1-\check{q}\check{q}^T)(\check{r}\check{r}^T + \check{s}\check{s}^T)\delta\check{p} = \check{o}\check{o}^T\delta\check{p}\,, \qquad (34)$$

where $\check{o}\in Q_b$, and $\check{o} = (-\check{r}+\kappa\check{s})/\sqrt{1+\kappa^2} = \check{h}\otimes\check{q}$. □

A similar but tedious derivation in [28] yields the following theorem for noise in the vector measurement b. We shall reuse some of the previous notation leading to theorem 11 and equation (32).

Theorem 12 *In the absence of any other errors, a perturbation error $\delta\check{b}$ in the vector measurement $\check{b}$ leads to a perturbation in the vector-aligned attitude estimate $\check{q}$ (equation (23)) equal to a rotation through the angle $-b\times\delta b$, which is the smallest angle rotation that takes b to $b+\delta b$.*

Proof: Taking differentials of equation (32):

$$\left.\begin{array}{r} \sqrt{2(\alpha^2+\beta^2)(1+b_3)}\delta\check{q} \\ +\check{q}\sqrt{2(1+b_3)}\dfrac{\alpha\delta\alpha+\beta\delta\beta}{\sqrt{\alpha^2+\beta^2}} \\ +\check{q}\sqrt{2(\alpha^2+\beta^2)}\dfrac{\delta b_3}{2\sqrt{1+b_3}} \end{array}\right\} = \left\{\begin{array}{r} \delta\alpha\check{u} + \delta\beta\check{v} \\ +\alpha\delta\check{u} + \beta\delta\check{v} \end{array}\right.. \qquad (35)$$

Similar to the proof of theorem 11, the coefficients multiplying the first order differentials are approximated to their nominal values, ultimately yielding

$$\delta\check{q} = -\frac{1}{2}\check{q}\otimes\check{b}\otimes\delta\check{b} = -\frac{1}{2}\check{o}\otimes\delta\check{b}\,, \qquad (36)$$

where $\breve{o} = \breve{h} \otimes \breve{q} = [(-q_3)\ (-q_2)\ q_1\ q_0]^T = \breve{q} \otimes \breve{b}$, and $\breve{q} \otimes \breve{b}$ is already orthogonal to $\breve{q}$. $\square$

Equations (16), (34), (36) can be used to derive an equation for the evolution of noise in the integrated and vector-aligned estimates.

$$\delta\breve{p}_{i+1} = \delta\breve{q}_i \otimes \left(\breve{1} + \frac{\breve{\omega}_i T}{2}\right) + \breve{q}_i \otimes \frac{\delta\breve{\omega}_i T}{2} = P_{i+1} \begin{bmatrix} \delta\breve{q}_i \\ \delta\omega_i \end{bmatrix},$$

$$\Rightarrow \delta\breve{q}_{i+1} = \breve{o}_{i+1}\breve{o}_{i+1}^T \left[\delta\breve{q}_i \otimes \left(\breve{1} + \frac{\breve{\omega}_i T}{2}\right) + \breve{q}_i \otimes \frac{\delta\breve{\omega}_i T}{2}\right]$$

$$- \frac{1}{2}\breve{o}_{i+1} \otimes \delta\breve{b}_{i+1} = Q_{i+1} \begin{bmatrix} \delta\breve{q}_i \\ \delta\omega_i \\ \delta b_{i+1} \end{bmatrix}, \quad (37)$$

where $\delta\breve{q}_i$ is the noise in the attitude estimate at the previous time-step. Equation (37) can be used to derive expressions for the covariance matrices corresponding to $\breve{p}$ and $\breve{q}$, Π and Ξ:

$$\Pi_{i+1} = P_{i+1} \begin{bmatrix} \Xi_i & \\ & W_i \end{bmatrix} P_{i+1}^T,$$

$$\Xi_{i+1} = Q_{i+1} \begin{bmatrix} \Xi_i & & \\ & W_i & \\ & & B_{f,i+1} \end{bmatrix} Q_{i+1}^T, \quad (38)$$

where Ξ, W, and B_f are the covariance matrices corresponding to the attitude estimate $\breve{q}$, angular velocity measurement noise $\delta\omega$, and filtered vector measurement noise δb_f respectively.

5 Simulation results

In this section, we use Matlab simulations to verify the key theoretical results derived in the previous section. The first group of simulations correspond to verifying the solution for the first problem – attitude estimation using two vector measurements. We assume that the directions of two linearly independent vectors, h and k, are measured at 100Hz in the body-fixed coordinate system as a and b. Measurements a and b are assumed to have random, unbiased noise of 0.01 and 0.02 normalized units respectively. The body is prescribed an oscillatory roll and pitch motion, and a constant yaw angle.

Figure 6, left, shows the attitude estimated using theorem 7, $\breve{q}_G$, in comparison with the attitude derived by using the TRIAD method, $\breve{q}_T$, when reference vector h is of greater significance. The attitude quaternions have been converted into Euler angles for ease of interpretation. Both the solutions are identical upto machine precision. The overlaying plots of the attitude $(\phi_T, \theta_T, \psi_T)$ show that the two solutions maintain equivalence even while the attitude follows a high-amplitude trajectory and the estimation errors are significant.

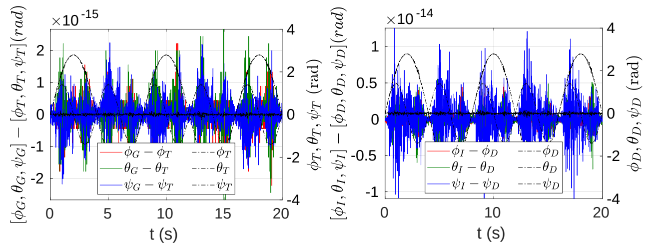

Fig. 6. Matlab simulations of full attitude estimation using two vector measurements. Left: This figure shows the results of applying the TRIAD solution and using the geometric method of Theorem 7. The figure shows that the two solutions are equal upto machine precision. Right: This figure shows the results of applying Davenport's q-method and an appropriate geometric filter using (11). The figure shows that the two solutions are equal upto machine precision.

By using equation (11) to interpolate between the two solutions obtained from theorem 7, we obtain the solution to Wahba's problem. The interpolation parameter x is chosen to be $2^2/(1^2 + 2^2) = 0.8$, as the rms noise of the two vector measurements have a ratio of 2. Figure 6 (right) shows the equivalence between the result obtained by interpolating (equation (11)) on the two estimates of theorem 7, $\breve{q}_I$, and that obtained by using Davenport's q-method, $\breve{q}_D$. The overlaying plots of the attitude $(\phi_D, \theta_D, \psi_D)$ show that the two solutions maintain equivalence even while the attitude follows a high-amplitude trajectory and the estimation errors are significant.

The next group of simulations verify the result of Theorem 9, and Remarks 9.2 and 10.1.

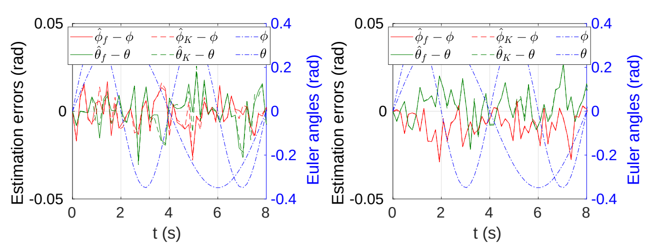

Fig. 7. Filtering using equations (30), (31), (38) to obtain a filtered attitude estimate. The roll and pitch angles are prescribed to be sinusoids of amplitude $\pi/9$ rad. Left: The filtered solution has lower errors than an optimally tuned EKF for large attitude corrections (≈ 0.16 units rms noise in angular velocity measurement) at each time-step. Right: In the limit of smaller attitude corrections (≈ 0.04 units rms noise), the EKF approaches the more accurate interpolated solution using equations (30), (31), (38).

The attitude estimate $\breve{q}$ of Theorem 9 can be filtered to reduce the noise, as decribed in equations (30), (31), (38). For small noise in the angular velocity measurement and consequently small attitude corrections at each time-step, the filtered estimate ($\hat{\phi}_f$, $\hat{\theta}_f$) is the same as that obtained using the traditional EKF ($\hat{\phi}_M$, $\hat{\theta}_M$), but the linearization inherent in the EKF begins to introduce significant errors for large corrections (figure 7). On the left, the rms noise in the angular velocity measurement is 0.16 units, while it is 0.04 units on the right. While the variance of the error is similar with both the methods on the right (0.98e-4 sq-units with the EKF and 0.96e-4 sq-units with the geometric filter), it is 21% lower with the geometric filter on the left (1.14e-4 sq-units with the EKF and 0.94e-4 sq-units with the geometric filter).

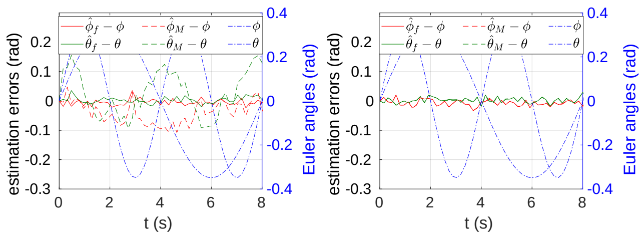

Fig. 8. A comparison of the estimator in Theorem 9 against the ECF in [21]. The ECF estimate ($\hat{\phi}_M$, $\hat{\theta}_M$) has larger residual errors unless we use the optimal gain suggested in this paper in a two-step estimation. Left: The ECF with gains recommended in [21]. Right: the ECF using the gain derived in Remark 10.1 in two-step estimation.

The attitude estimator in Theorem 9 ($\hat{\phi}_f$ and $\hat{\theta}_f$) is compared with the ECF of [21] ($\hat{\phi}_M$ and $\hat{\theta}_M$) in figure 8. The true attitude angles are denoted ϕ and θ. The geometric filter provides superior accuracy to the ECF with the gains recommended in [21]. Equivalent performance may be obtained with both the solutions only upon following a two-step attitude estimation in the ECF, and using the gains suggested in Remark 10.1. The two-step estimation is essential so as to ensure that the angular velocity correction ω_c is with respect to the filtered vector measurement b_f obtained from the first step, and that the subsequent vector-measurement based correction is expressed in the body-frame obtained after integrating the angular velocity in the first step.

6 Experimental validation of geometric attitude estimation using rate and single vector measurement

This section provides experimental verification for the geometric attitude estimator by using a recently developed autopilot in our group, which is equipped with an IMU, the MPU9250, and is described in [1]. The autopilot is mounted on an inhouse designed model positioning system (MPS) that can independently prescribe roll, pitch, plunge and yaw manoeuvres on a test module.

The roll motion has an amplitude of $5\pi/6$ and a period of 4s. The pitch motion has the same period, and an amplitude of $4\pi/9$. The encoder on the MPS provides the true angles at 1kHz, while the attitude estimator

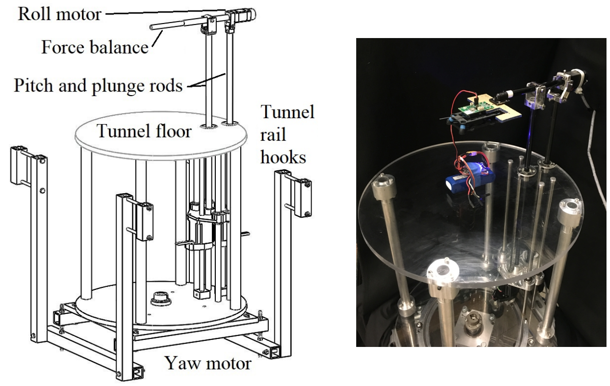

Fig. 9. On the left, a schematic of the 4 Degree of freedom Model Positioning System (MPS) described in [23]. The MPU9250 mounted on the PCB (green in the picture on the right) and being tested on the MPS.

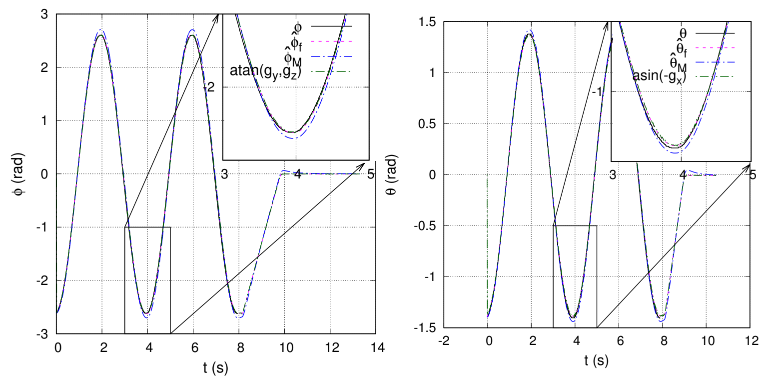

Fig. 10. Left: Attitude estimation for a pure sinusoidal roll manoeuvre on a real system. Right: Attitude estimation for a pure sinusoidal pitch manoeuvre on a real system. The solid black lines are the true roll and pitch angles returned by the encoder, and the dashed magenta curves are their estimates using Theorem 9 presented in this paper after the filtering described in section 4. The dash-dot blue curve shows the attitude estimate obtained using the ECF [21]. The dash–dot-dot green line is the attitude consistent with the gravity vector measurement.

on the MPU9250 provides estimates at 90Hz. The estimated roll and pitch angles are plotted along with the true values in figure 10. The residual errors in estimating the roll and pitch angles can be attributed to experimental errors. Also shown in the zoomed insets is the high-accuracy, zero latency tracking from the vector measurements to the attitude estimation. This may be compared with the larger errors using the ECF. As shown in Remark 10.1, the ECF is an approximation of the exact geometric estimation that is associated with latency on account of a feedback based correction mechanism. In this experiment, the ECF was used with a gain k_P equal to 1, as suggested in [21]. Using lower values for k_P introduces greater latency for a gradual worsening in the asymptotic accuracy.

7 Conclusion

We have reported a geometry-based analytic solution for the problem of attitude estimation using two reference vector measurements, and using a rate measurement and a measurement of a single reference vector. The estimated attitude is analytically derived, so that the need to tune gains does not arise. The estimate also has no latency and is available at the same timestep when the measurement is available. The estimator is verified using Matlab simulations and also by experiments for accuracy and responsiveness.

The presented approach also leads to a unified framework to derive, as special cases, the most significant among previously reported solutions: namely, the TRIAD solution [8], Wahba's formulation [7], the extended Kalman filter [5], and the ECF [21]. These four works represent the four most common approaches for attitude estimation: the former two for estimation using vector observations, the EKF for estimation using a linearized complementary filter, the ECF for estimation using a nonlinear complementary filter. Beyond the optimality metrics of these formulations, the proposed solution can also handle nonlinear and non-holonomic optimization.

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