Compensation of measurement noise and bias in geometric attitude estimation*

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Abstract— **A geometry-based analytic attitude estimation using a rate measurement and measurement of a single reference vector has been recently proposed. Because rigid body attitude estimation is a fundamentally nonlinear problem, the geometry-based method does not contain errors consequent to linearization approximations. A critical source of residual error in the geometric solution is on account of the noise and bias in the vector and rate measurements. A methodical perturbation analysis of the attitude estimate is performed in this paper that reveals the effects of measurement noise and bias, and provides means to compensate for, or filter out, such errors. Application of the filter and compensation provides better attitude estimation than a standard Extended Kalman filter using an optimal Kalman gain. The geometric method is first verified in experiments and then simulation results are provided that validate the better performance of the geometric attitude and bias estimator.**

I. Introduction

Recent work by the authors [1] has introduced a geometry-based analytic solution for the single vector measurement attitude estimation problem. In contrast to existing work [2]–[6], the geometric solution presents an instantaneous estimate that is heedful of the nonlinearity of the attitude dynamics in the absence of measurement errors. While the Extended Kalman filter (EKF) solution in [2] is also a point-wise estimator that returns atitude estimates with no latency, it is associated with the errors consequent to the linearization of the attitude dynamics and constraints associated with angular velocity and vector measurements. On the other hand, filter-based solutions (see [3]–[6]) all use negative feedback upon an existing estimation error, and show latency in the attitude estimation with respect to the measurements.

Given the advantages of the geometric method in the absence of measurement errors, the issue in order is then to analyze the effect of measurement errors on the attitude estimation, and then to remedy the effect. One may distinguish between two kinds of errors: (i) zero-mean Gaussian noise, that is bandlimited to a frequency proportional to the sensor sampling bandwidth, (ii) an Ornstein-Uhlenbeck process with a non-zero mean, that has exponential autocorrelation. The former kind of error is associated with both the vector as well as angular velocity measurements. The latter error is most often associated with the angular velocity measurement, and is commonly referred to as a gyroscopic bias error.

The band-limited Gaussian noise may be filtered at individual time steps in order to obtain a minimum mean square error (MMSE) attitude estimate. The filter gains are typically functions of the state and the noise covariance matrices and complementarily weigh the angular velocity and vector measurements in order to provide an optimal estimate. It may be noted that this noise is assumed to be band-limited in order to prevent aliasing upon sampling the sensor in discrete time measurements. Consequently, the root mean square noise is proportional to the square root of the sampling frequency.

In contrast to the Gausian noise, the gyroscopic bias error varies slowly with time. It may be estimated from the history of attitude corrections over time that result from aligning the integrated attitude estimates to the vector observations. Since the bias is assumed to be exponentially autocorrelated with a time constant much larger than the time-step between measurements, this filter is cumulative with respect to the corrections, and does not neglect past corrections. In fact, consideration of past corrections is necessary, because the attitude correction at a single time-step covers only the two-dimensional subspace orthogonal to the vector observation and this makes the three-dimensional bias unobservable.

The estimation of a constant or slowly varying gyroscopic bias has been reported in [7]–[11]. Since these works were based upon an observer-based attitude estimate, the gyroscopic bias is also estimated using an observer with a tuned gain. In contrast, the geometric attitude estimator enables the design of a bias estimator without gain tuning, with the "gains" arising naturally out of the properties of the measurement errors. Moreover, the observability condition for the bias estimator can be expressed as a simple condition on the vector measurements.

A brief outline of this paper is as follows. The next section (section II) introduces various definitions, and notation associated with the problem, and presents a precise mathematical problem statement. Section III considers the effects of the first kind of measurement noise (Gaussian delta-autocorrelated) on the attitude estimate, and describes how the noise may be filtered. Section IV considers effects of the nonzero-mean, exponentially auto-correlated bias, and describes a method to estimate and compensate for the bias.

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II. Notation, definitions, and problem statement

We shall use unit quaternions $\breve{p}, \breve{q}, \ldots$ to denote rigid body attitude with respect to a reference coordinate system [12]. Quaternion multiplication signifying the composition of active body rotations $\breve{p}$, and subsequently $\breve{q}$, shall be written as $\breve{q} \otimes \breve{p}$ in the reference coordinate system, and $\breve{p} \otimes \breve{q}$ in the body-fixed coordinate system.

Pertinent to the work in this paper is the problem of sequentially estimating the attitude quaternion $\breve{q}$ using a sequence of vector and angular velocity measurements, $\hat{b}$ and $\hat{\omega}$, starting from a known initial attitude (without loss of generality, the identity quaternion $\breve{1}$).

The kinematic equation in terms of the angular velocity ω provides an equation for the incremental change in the attitude with respect to the estimate at a previous time-step, in order to derive a kinematic estimate $\breve{p}$:

$$\breve{p}_{i+1} = \breve{q}_i + \frac{T}{2} \breve{q}_i \otimes \breve{\omega}, \tag{1}$$

where T is a small integration time-step, $\breve{\omega} = [0 \; \omega^T]^T$ is the quaternion corresponding to the angular velocity 3-vector ω. If there were no errors in the angular velocity measurement, $\breve{q}_{i+1}$ would be equal to $\breve{p}_{i+1}$, and an integration of equation (1) would yield the attitude estimate $\breve{q}$ at any desired time-step. Since there is no ambiguity, we shall drop the usage of subscripts $i, \ldots$ to denote the time-step until the latter portion of section III.

The angular velocity of the body is measured to have components $\hat{\omega} = [\hat{\omega}_1 \;\; \hat{\omega}_2 \;\; \hat{\omega}_3]^T$ in the body coordinate system. This is the typical scenario in most applications, where the gyroscope is part of an Inertial Measurement Unit (IMU) that is fixed with respect to the body. However, the measured angular velocity $\hat{\omega}$ has an error with respect to the true quantity ω. The angular velocity measurement error is assumed to be an Ornstein-Uhlenbeck process, with mean $\overline{\omega}$, time-constant τ, and random-walk increments $\tilde{\omega}$:

$$\hat{\omega} = \omega + \overline{\omega} + \tilde{\omega}. \tag{2}$$

A reference vector $\mathbf{h}$ is any given direction of unit magnitude in the reference coordinate system. Its components, $h = [h_1 \;\; h_2 \;\; h_3]^T$ in the reference coordinate system, and $b = [b_1 \;\; b_2 \;\; b_3]^T$ in the body-fixed coordinate system, are related by the rotational transformation corresponding to the body's attitude:

$$\breve{q} \otimes \breve{b} = \breve{h} \otimes \breve{q}, \tag{3}$$

where $\breve{h} = [0 \;\; h^T]^T$ and $\breve{b} = [0 \;\; b^T]^T$ are the quaternions corresponding to the 3-vectors h and b. This provides a second equation for the attitude, $\breve{q}$, constraining it to lie on a single-dimensional "feasibility cone", Q_b [1].

Again, the measurement $\hat{b}$ of b is expected to have an error. In the case of the vector measurement, the error $\tilde{b}$ is assumed to be a zero-mean Gaussian process.

$$\hat{b} = b + \tilde{b}. \tag{4}$$

Since the sensor noise does not preserve the vector's unit norm, we shall assume that $\hat{b}$ is explicitly normalized before being passed on to the estimator, so that, for small $\tilde{b}$, $b^T \tilde{b} = 0$.

The problem we pose is then *to estimate the bias $\overline{\omega} + \tilde{\omega}$ in the angular velocity measurement $\hat{\omega}$, and filter the residual Gaussian noise in $\tilde{\omega}$ and $\tilde{b}$, in order to obtain an optimal estimate $\breve{q}$ for the rigid body attitude using a time-sequence of angular velocity measurements $\hat{\omega}$ and reference vector measurements $\hat{b}$.*

III. Effect of small signal measurement noise

We first analyze the effect of unbiased (zero-mean), Gaussian noise in the angular velocity measurement $\hat{\omega}$ and vector measurement $\hat{b}$ on the estimated attitude $\breve{q}$. In particular, we shall assume that there is no random-walk or constant nonzero bias error in $\hat{\omega}$. Further, we shall make the reasonable assumption that the Gaussian noise is small enough relative to the norms of the quantities to consider them as perturbations, and therefore add the effects of individual noise sources to obtain the cumulative effect.

We shall introduce some new notation, to avoid lengthy expressions. When the reference z-axis is aligned along the vector $\mathbf{h}$, the quaternion attitude estimate is given by [1]:

$$\sqrt{2(\alpha^2 + \beta^2)(1 + b_3)}\, \breve{q} = \begin{bmatrix} \alpha(1 + b_3) \\ \alpha b_2 + \beta b_1 \\ -\alpha b_1 + \beta b_2 \\ \beta(1 + b_3) \end{bmatrix} = \alpha \breve{u} + \beta \breve{v}\,, \tag{5}$$

where $\breve{u} = [(1 + b_3) \;\; b_2 \;\; -b_1 \;\; 0]^T$ and $\breve{v} = [0 \;\; b_1 \;\; b_2 \;\; (1 + b_3)]^T$ are an orthogonal basis for the feasibility cone Q_b corresponding to the vector measurement $\hat{b}$, $\alpha = p_0(1+b_3) + p_1 b_2 - p_2 b_1 = \breve{p}^T \breve{u}$, $\beta = p_1 b_1 + p_2 b_2 + p_3(1 + b_3) = \breve{p}^T \breve{v}$, and $\breve{p} = [p_0 \;\; p_1 \;\; p_2 \;\; p_3]^T$ is the integrated attitude estimate obtained using equation (1) on the best estimate of the previous time-step.

The effect on $\breve{q}$ of a perturbation $\delta\breve{p}$ in the integrated estimate $\breve{p}$ is shown in [1] to be given by:

$$\delta\breve{q} = \breve{o}\breve{o}^T \delta\breve{p}\,, \tag{6}$$

where $\breve{o} \in Q_b$, and $\breve{o} = (-\beta\breve{u} + \alpha\breve{v})/\| - \beta\breve{u} + \alpha\breve{v}\| = \breve{h} \otimes \breve{q}$. Thus, in the absence of any other errors, a perturbation error $\delta\breve{p}$ in the integrated attitude estimate $\breve{p}$ leads to a perturbation in the vector-aligned attitude estimate $\breve{q}$ (equation (5)) equal to the projection of $\delta\breve{p}$ onto the feasibility cone orthogonal to $\breve{q}$.

A similar but tedious derivation yields the following result for noise in the vector measurement $\hat{b}$:

$$\delta\breve{q} = -\frac{1}{2} \breve{q} \otimes \breve{b} \otimes \delta\breve{b} = -\frac{1}{2} \breve{o} \otimes \delta\breve{b}\,, \tag{7}$$

where $\breve{o} = \breve{h} \otimes \breve{q} = [(-q_3) \;\; (-q_2) \;\; q_1 \;\; q_0]^T = \breve{q} \otimes \breve{b}$, and $\breve{q} \otimes \breve{b}$ is already orthogonal to $\breve{q}$. In the absence of any other errors, a perturbation error $\delta\breve{b}$ in the vector measurement $\breve{b}$ thus leads to a perturbation in the vector-aligned attitude estimate $\breve{q}$ (equation (5)) equal to a rotation through the angle $-b \times \delta b$, which is the smallest angle rotation that takes b to $b + \delta b$.

Equations (1, 6, 7) can be used to derive an equation for the evolution of noise in the integrated and vector-aligned estimates:

$$\delta\breve{p}_{i+1} = \delta\breve{q}_i \otimes \left(\breve{1} + \frac{\breve{\omega}_i T}{2}\right) + \breve{q}_i \otimes \frac{\delta\breve{\omega}_i T}{2} = P_{i+1}\begin{bmatrix}\delta\breve{q}_i \\ \delta\omega_i\end{bmatrix},$$

$$\Rightarrow \delta\breve{q}_{i+1} = \breve{o}_{i+1}\breve{o}_{i+1}^T\left[\delta\breve{q}_i \otimes \left(\breve{1} + \frac{\breve{\omega}_i T}{2}\right) + \breve{q}_i \otimes \frac{\delta\breve{\omega}_i T}{2}\right]$$
$$- \frac{1}{2}\breve{o}_{i+1} \otimes \delta\breve{b}_{i+1} = Q_{i+1}\begin{bmatrix}\delta\breve{q}_i \\ \delta\omega_i \\ \delta b_{i+1}\end{bmatrix}, \qquad (8)$$

where $\delta\breve{q}_i$ is the noise in the attitude estimate at the previous time-step. Equation (8) can be used to derive expressions for the covariance matrices corresponding to $\breve{p}$ and $\breve{q}$, say Π and Ξ:

$$\Pi_{i+1} = P_{i+1}\begin{bmatrix}\Xi_i & \\ & W_i\end{bmatrix}P_{i+1}^T,$$
$$\Xi_{i+1} = Q_{i+1}\begin{bmatrix}\Xi_i & & \\ & W_i & \\ & & B_{i+1}\end{bmatrix}Q_{i+1}^T, \qquad (9)$$

where Ξ, W, and B are the covariance matrices corresponding to the attitude estimate $\breve{q}$, measurement noise $\delta\omega$, and δb respectively.

It may be noted at this point that the errors in $\breve{p}$ and $\breve{q}$ are not completely independent. The common portion of the errors arises on account of equation (6), which shows that the projection of $\delta\breve{p}$ on $\breve{o}$, $\breve{o}\breve{o}^T\delta\breve{p}$, also appears in $\delta\breve{q}$, and cannot be filtered out. Another way to see this is by noting that the vector observation $\hat{b}$ provided no information for the attitude estimate within its feasibility cone Q_b, and the feasibility cone is accessed from $\breve{q}$ along $\breve{o}$. This common error $\breve{o}\breve{o}^T\delta\breve{p}$, the orthogonal complement of $\delta\breve{p}$ with respect to $\breve{o}$, $(1-\breve{o}\breve{o}^T)\delta\breve{p}$, and the error in $\breve{q}$ on account of δb, $-\breve{o}\otimes\delta\breve{b}/2$, are three independent errors in the two attitude estimates, $\breve{p}$ and $\breve{q}$. Of these, the latter two may be complementarily filtered in order to obtain an unbiased filtered estimate $\breve{q}_f$:

$$\breve{q}_{f,i+1} = (\Sigma_{i+1}^{-1} + B_{i+1}^{-1})^{-1}(\Sigma_{i+1}^{-1}\breve{p}_{i+1} + B_{i+1}^{-1}\breve{q}_{i+1}), \quad (10)$$

where, $\Sigma = (1-\breve{o}\breve{o}^T)\Pi(1-\breve{o}\breve{o}^T)$ is the covariance matrix of $(1-\breve{o}\breve{o}^T)\breve{p}$.

IV. Observability and Estimation of Gyroscopic Bias

In this section, we consider the effects of gyroscopic bias on the geometric attitude estimation. Since the gyroscopic bias is exponentially autocorrelated with a time constant that is much larger than the time-step between measurements, this error manifests as a relatively low frequency source in comparison to the Gaussian noise considered in the previous section. The slow variation with time enables the design of an observer that could estimate the noise as well as compensate for it.

First, consider a bias error $\overline{\omega}$ that is constant with time. At each time-step, the estimate obtained by integrating the angular velocity is projected onto the feasibility cone corresponding to the vector measurement. The incremental change from the integrated attitude quaternion estimate, $\breve{p}$, to the vector-aligned estimate, $\breve{q}$, is essentially the correction to the integrated error in the angular velocity measurement $\overline{\omega}$. Denoting the increment by $\breve{r}$ in the body-fixed coordinate system (since $\hat{\omega}$ is available only in this system), for a constant $\overline{\omega}$ over a small integration time T, we must have [1]:

$$\breve{p}^{-1} \otimes \breve{q} \approx \breve{1} - \frac{T\breve{\overline{\omega}}}{2} = \breve{r} = \begin{bmatrix}1 \\ -\overline{\omega}T/2\end{bmatrix}$$
$$\breve{r} = \begin{bmatrix}1 \\ \delta r\end{bmatrix} = \breve{p}^{-1} \otimes \breve{q} = \begin{bmatrix}1 \\ -\overline{\omega}T/2\end{bmatrix} + \delta\mu\breve{b}, \qquad (11)$$

where $\delta\mu$ is an unknown infinitesimal rotation about $\hat{b}$ in the body system. We have assumed that we start on the feasibility cone, and integrate the angular velocity measurement over a small time, so $\breve{r}$ is close to $\breve{1}$, and its scalar portion is approximately 1. However, with a single vector measurement, a correction is possible only in the subspace orthogonal to the measured vector $\hat{b}$. Therefore, we have an unknown term proportional to $\breve{b}$ in equation (11). Projecting onto the subspace orthogonal to $\hat{b}$, we obtain $(1-\hat{b}\hat{b}^T)\overline{\omega} = 2(\hat{b}\hat{b}^T - 1)\delta r/T$ in the case of a correction onto the feasibility cone of a single measurement $\hat{b}$. Since $\hat{b}$ and δr are known, this may be used to estimate the portion of $\overline{\omega}$ normal to $\hat{b}$. With two or more linearly independent measurements $\hat{b}_j$ and corrections δr_j at a constant $\overline{\omega}$, the matrix $\sum_j(1-\hat{b}_j\hat{b}_j^T)$ becomes invertible, and we can actually determine $\overline{\omega}$ completely:

$$\sum_j(1-\hat{b}_j\hat{b}_j^T)\overline{\omega} = \sum_j 2(\hat{b}_j\hat{b}_j^T - 1)\delta r_j/T\,. \qquad (12)$$

Thus, in the absence of any other measurement errors, a fixed bias error in the angular velocity measurement can be completely estimated using equation (12) on two linearly independent vector measurements.

The condition for invertibility of $\sum_j(1-\hat{b}_j\hat{b}_j^T)$ is the same as the full-rank condition in literature, and for sequential measurements of a single vector, it is equivalent to the persistently non-parallel and sufficient excitation conditions. The condition can easily be checked by evaluating $\hat{b}_i^T\sum_{j=1}^i(1-\hat{b}_j\hat{b}_j^T)$, since each of the terms in the summation is positive semi definite, and the inner product of $\hat{b}_i$ with the last term returns zero. Therefore, the summation is invertible if and only if its inner product with $\hat{b}_i$ is non-zero.

Let us now allow variation in the bias error through the $\tilde{\omega}$ term (equation (2)). If only measurements of a single constant vector are available, the variation in $\tilde{\omega} = \hat{\omega} - \omega - \overline{\omega}$ can cause the bias estimation of equation (12) to be inaccurate. In this case of time-varying bias $\tilde{\omega}$, we would be estimating the weighted average of the error, $\overline{\omega} + \tilde{\omega}$, during the time over which the measurements were taken and the corrections determined. For a uniformly distributed attitude, that would just be the constant bias error $\overline{\omega}$ in equation (12). Equation (12) assigns equal weightage to all past measurements and corrections. This suggests a mechanism for estimating a slowly varying bias. Rather than weigh all past measurements equally, their influence on the current bias estimation may be progressively and gradually reduced (analogous to an infinite impulse response filter). This simulates a low pass filter on the attitude corrections whose bandwidth may be determined by the time constant τ of the autocorrelation of the bias error. For *e.g.*, if $\tau/T = 100$, then we could reduce the influence of past measurements by $1 - 1/100 = 0.99$ in each successive measurement. Increasing the attenuation factor towards 1 reduces the bandwidth of the bias estimator and lowers the noise in the estimation. Contrarily, reducing the attenuation factor towards 0 increases the bandwidth of the bias estimator, but at the cost of higher noise. Such an estimator may be expressed in terms of the matrices A_i and B_i, defined inductively, as shown below:

$$\begin{aligned} A_{i+1} &= (1 - T/\tau)A_i + (T/\tau)(1 - \hat{b}_i\hat{b}_i^T), \\ B_{i+1} &= (1 - T/\tau)B_i + (\hat{b}_i\hat{b}_i^T - 1)2\delta r_i/\tau, \\ A_{i+1}(\overline{\omega} &+ \tilde{\omega}_{i+1}) = B_{i+1}, \end{aligned} \tag{13}$$

with the initial conditions $A_0 = 0, B_0 = 0$. Note that $\overline{\omega}$ is a constant across the time-steps, and is the output of the estimator in the special limiting case when τ goes to infinity.

While equation (13) is sufficient to estimate the bias when the persistency-of-excitation condition is met, it may fail when the body stops rotating. The failure upon loss of excitation occurs as $\hat{b}_i$ approaches a limit, and the matrix A_i gradually approaches the now constant $1 - \hat{b}_i\hat{b}_i^T$ over time, thus becoming singular. Failure may be avoided under such circumstances by updating only the components of A_i and B_i that have additional information in the new measurements, as done in the following estimator design:

$$\begin{aligned} A_{i+1} &= (\hat{b}_i\hat{b}_i^T)A_i \\ &\quad + (1 - \hat{b}_i\hat{b}_i^T)((1 - T/\tau)A_i + (T/\tau)), \\ B_{i+1} &= (\hat{b}_i\hat{b}_i^T)B_i \\ &\quad + (1 - \hat{b}_i\hat{b}_i^T)((1 - T/\tau)B_i - 2\delta r_i/\tau), \\ A_{i+1}&(\overline{\omega} + \tilde{\omega}_{i+1}) = B_{i+1}, \end{aligned} \tag{14}$$

The estimator of equation (14) tracks a time-varying bias equally as well as that in equation (13) under persistant excitation. However, it does not fail when excitation is lost. It provides the best estimate of the bias it could under the circumstances: tracking the components of the bias orthogonal to $\hat{b}_i$, while retaining the last best estimate for the component of bias along $\hat{b}_i$. The first order filtering can easily be extended to higher orders by including additional terms on the right hand side of equation (14) that invoke A_{i-1}, B_{i-1} *etc.*

V. Simulation and Experimental Results

We first verify the geometric attitude estimator experimentally by using a recently developed autopilot in our group, which is equipped with an IMU, the MPU9250, and is described in [13]. The autopilot is mounted on an inhouse designed model positioning system (MPS) that can independently prescribe roll, pitch, plunge and yaw manoeuvres on a test module. The 4 degree-of-freedom MPS is described in [14]. A key enabling feature of the MPS is that it provides for both static and dynamic positioning of a mounted model, which is required to generate and measure a non-zero angular velocity.

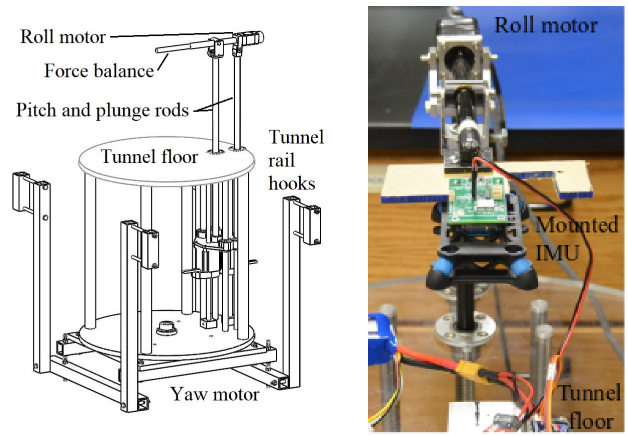

Fig. 1: On the left, a schematic of the 4 Degree of freedom Model Positioning System (MPS) described in [14]. The MPU9250 mounted on the PCB (green in the picture on the right) and being tested on the MPS.

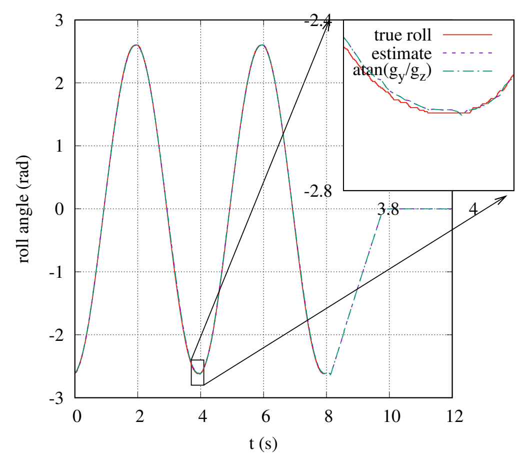

Fig. 2: Attitude estimation for a pure sinusoidal roll manoeuvre on a real system. The solid red line is the true roll angle returned by the encoder, and the dashed blue curve is the estimated roll angle. The small residual estimation errors can be mostly traced to alignment inaccuracies of the testbed. The vector measurements, g_y and g_z, are also included to show the instantaneous response from the measurement to the estimation.

The roll and yaw motions are generated using stepper motors which can be programmed to rotate the model according to a prescribed trajectory. As the motors rotate, real-time measurement is provided using 517 counts-per-revolution differential encoders which provide feedback to the motor controller and also a record of the actual position of the tested model. Pitch and plunge motion are generated

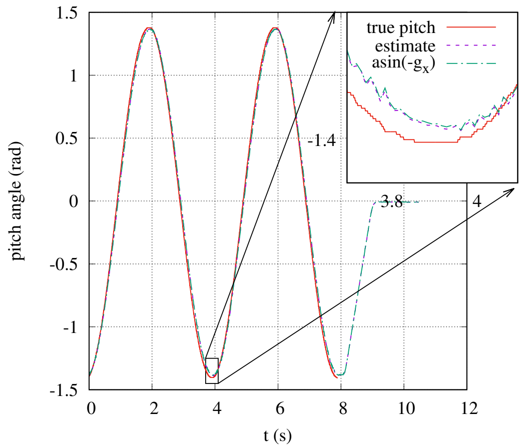

Fig. 3: Attitude estimation for a pure sinusoidal pitch manoeuvre on a real system. The solid red line is the true pitch angle returned by the encoder, and the dashed blue curve is the estimated pitch angle. The small residual estimation errors can be mostly traced to alignment inaccuracies of the testbed. The vector measurement, g_x, is also included to show the instantaneous response from the measurement to the estimation.

by actuating the pitch plunge rods using linear accelerators. The output of the rods is again measured using encoders to provide real-time feedback and a record of the actual position of the model. During all the tests, the IMU is positioned on the rotation axis so that there is no acceleration, and the accelerometer is sensing only the gravitational field.

The autopilot is then separately subjected to oscillatory roll and pitch motions. The roll motion has an amplitude of $5\pi/6$ and a period of 4s. The pitch motion has the same period, and an amplitude of $4\pi/9$. The estimated roll and pitch angles are plotted along with the true values in figures 2 and 3. The residual errors in estimating the roll and pitch angles can be attributed to experimental errors. Also shown in the zoomed insets is the zero latency tracking from the vector measurements to the attitude estimation.

We next verify the filtered estimate of equation (9) using Matlab simulations. For the simulations, the body is prescribed a sinusoidal pitch manoeuvre of amplitude $4\pi/9$ rad and period 4 s. The angular velocity and vector measurements are associated with a zero-mean Gaussian noise of 0.04 rad/s and 0.01 respectively. The attitude of the body is then estimated using an optimally tuned extended Kalman filter (figure 4) and by equation (9) (figure 5). It can be seen that for large attitude increments between time-steps, the linearized estimation using an EKF leads to loss of accuracy, and the geometric attitude etimator presents nearly three-fold lesser noise in the root-mean-square sense.

We finally verify the gyroscopic bias estimation using equation (13). The body is prescribed the same oscillatory roll motion as previous. However, the angular velocity measurement is now associated with a nonzero mean, random walk noise (an Ornstein-Uhlenbeck process, to be precise).

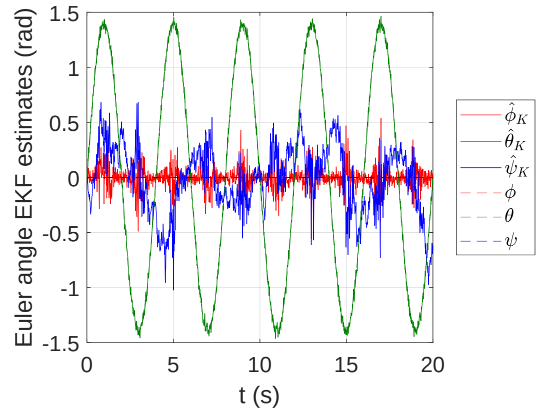

Fig. 4: Filtered estimation using an optimally tuned extended Kalman Filter (EKF). It can be seen that the EKF estimate $\hat{\phi}_K$, $\hat{\theta}_K$, and $\hat{\psi}_K$ has higher residual noise in comparison with figure 5, which can be traced to the linearization of the nonlinear attitude dynamics.

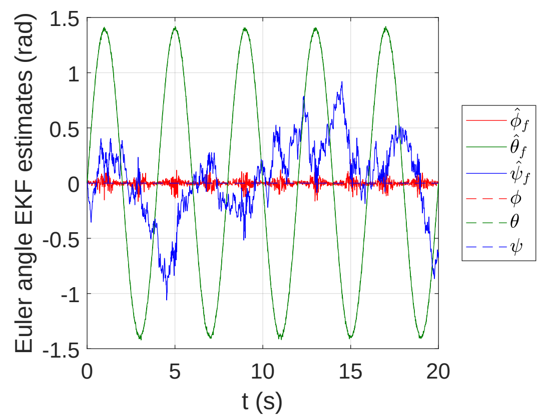

Fig. 5: Filtered estimation using equation (9). It can be seen that the proposed estimator yields more accurate estimates, $\hat{\phi}_f$, $\hat{\theta}_f$, and $\hat{\psi}_f$, than the optimally tuned EKF (*cf* figure 4).

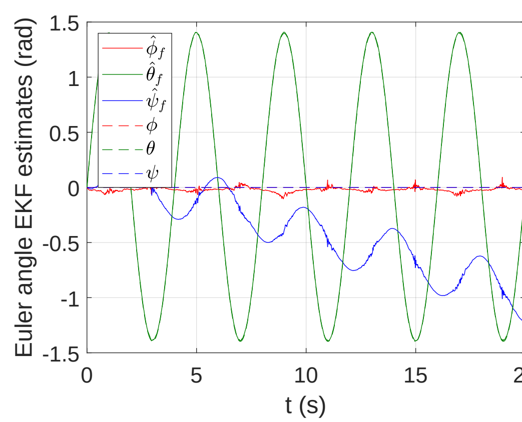

Fig. 6: Filtered attitude estimation in the presence of time-varying gyroscopic bias. Estimates $\hat{\phi}_f$, $\hat{\theta}_f$, and $\hat{\psi}_f$ of ϕ, θ, and ψ are obtained using equation (14) and used for bias estimation (figure 7). Notice the steady yaw-drift with bias.

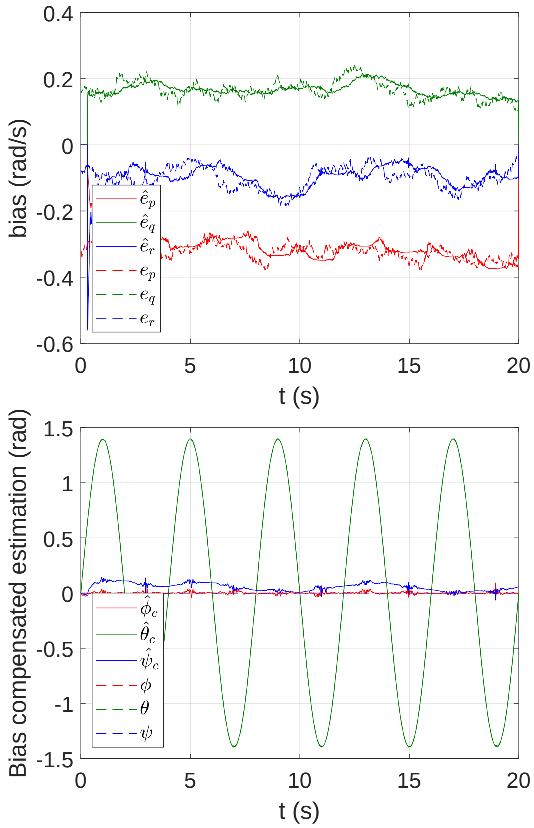

Fig. 7: Fail safe estimation of time-varying gyroscopic bias. A filtered attitude estimate is obtained using equation (14) (*cf* figure 6). The attitude correction from the integrated attitude estimate to the filtered estimate is inverted to obtain the bias error under persistent excitation (top). The filtered attitude is then corrected further to obtain compensated estimate, $\hat{\phi}_c$, $\hat{\theta}_c$, and $\hat{\psi}_c$, that gets rid of the low frequency errors in the bias (bottom).

An attitude is first estimated ($\hat{\phi}_f$, $\hat{\theta}_f$, $\hat{\psi}_f$) by filtering the high-frequency noise in the measurements (figure 6). This attitude is then used to derive the applied attitude correction from the integrated estimate of equation (1) to the filtered estimate in equation (5). The attitude correction over a sufficiently excited motion is then inverted to obtain an estimate $\hat{e}$ of the low frequency bias e in the gyroscope measurements (figure 7 top). The bias can then be compensated in the filtered estimate to obtain an asymptotically zero yaw drift (figure 7 bottom).

VI. Conclusion

Presented in this paper are an analysis of the effect of typical errors in vector and angular velocity measurements on attitude estimation, and methods to estimate, and compensate for, some of the errors. Specifically, constant or slowly varying bias in the angular velocity measurement can be estimated and compensated for, while high frequency Gaussian noise may be suitably filtered out. Of special note is the absence of adhoc gain tuning in the filter or estimator. The filter and estimator designs arise naturally out of the properties of the measurement errors. After validating the attitude estimator using experiments, simulations are used to verify the noise filter and bias estimator.

VII. Acknowledgements

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