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ON THE CONVERGENCE OF THE CONVECTIVELY FILTERED BURGERS EQUATION TO THE ENTROPY SOLUTION OF THE INVISCID BURGERS EQUATION[*]

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Abstract. This paper provides a proof that the solutions to the convectively filtered Burgers equation will converge to the entropy solution of the inviscid Burgers equation when certain restrictions are put on the initial conditions. It does so by first establishing convergence to a weak solution of the inviscid Burgers equation and then showing that the weak solution is the entropy solution. Then the results are extended to encompass more general initial conditions.

1. Introduction. Using a filtered velocity in fluid dynamics is not a new concept. Filtered velocities have been used in turbulence modeling in large eddy simulation (LES) [1, 2, 3], Lagrangian averaged Navier–Stokes (LANS-α) [4, 5, 6, 7], and Leray turbulence modeling [8, 9, 10]. Specifically in the LANS-α and Leray approaches, a filtered velocity is used in the nonlinear term of the Navier–Stokes equations. A form of the compressible Euler equations with a filtered velocity has also been developed using the Lagrangian averaging [11]. In our earlier paper [12], it was discussed that it should be possible to model both turbulence and shock formation using such a filtered velocity. This was motivated by realizing that turbulence and shocks are both consequences of the nonlinear term and its resulting cascade of energy into smaller scales. Thus it should be possible to capture both effects with proper small scale modeling. It has been seen that some turbulent behavior has been successfully modeled using a filtered velocity in the LANS-α and Leray approaches. This paper, in conjunction with our previous paper [12], aims at showing that such a technique can successfully model shock formation.

The investigation begins with the inviscid Burgers equation,

$$(1.1) \qquad\qquad u_t + uu_x = 0.$$

The Burgers equation was chosen because it shares the same nonlinear term as the Euler and Navier–Stokes equations. Additionally it is a conservation law, like the Euler equations. It is known to form shocks, and it has been well studied.

It is well established that the inviscid Burgers equation forms discontinuities in finite time, determined by initial conditions [13, 14]. To deal with these discontinuities, weak solutions are introduced. However, when weak solutions are introduced,

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solutions are no longer necessarily unique [14, 15]. In order to choose the physically relevant solution, an entropy condition is applied, which one and only one weak solution satisfies. This physically relevant solution is referred to as the entropy solution. Lax, Oleinik, and Kruzkov have examined the entropy condition for conservation laws and expressed it using different techniques [14, 15, 16]. Each of their entropy conditions can be used in different classes of conservation laws, but all can be applied to the inviscid Burgers equation with equivalent results [17]. This paper uses the Lax entropy condition, which is explained in section 2.

Classically the inviscid Burgers equation is regularized by adding viscosity, resulting in the equation

$$(1.2) \qquad u_t + u u_x = \nu u_{xx}.$$

This regularization has been proven to converge to the entropy solution of the inviscid Burgers equation as $\nu \to 0$ [14, 15, 16].

This paper considers the equations

$$(1.3a) \qquad u_t + \bar{u} u_x = 0,$$
$$(1.3b) \qquad \bar{u} = g^\alpha * u,$$
$$(1.3c) \qquad u(x,0) = u_0(x),$$

where

$$(1.4) \qquad g^\alpha = \frac{1}{\alpha} g\left(\frac{x}{\alpha}\right),$$

where g is a chosen filter. These equations replace the convective velocity of the inviscid Burgers equation with a filtered velocity. Thus, (1.3a) and (1.3b) are referred to as the convectively filtered Burgers (CFB) equation. While it has been proven that the solutions to the CFB equations exist [12], previously it has only been proven that the solutions for the Helmholtz filter converge to a weak solution of the inviscid Burgers equation with attempts to show numerically convergence to the entropy solution [18].

This paper proves that for a specific set of initial conditions the solutions to the CFB equations converge to the entropy solution of the inviscid Burgers equation. Specifically we will look at bell shaped, continuously differentiable initial conditions, rigorously defined in Definition 4.1. We then give a rationale and make a conjecture on how the CFB equations will converge to the entropy solution for any continuous initial conditions, and how to regain an entropy solution for discontinuous initial conditions.

The following section reviews established facts about the inviscid Burgers equation and some of the recent work regarding the CFB equations. Section 3 proves that solutions to the CFB equations converge to a weak solution of the inviscid Burgers equation, and section 4 proves convergence to the entropy solution. Section 5 then extends the results of section 4 and conjectures that they can be extended further. Section 6 runs some numerical simulations and examines the results. Section 7 follows with concluding remarks.

2. Background information on the Burgers equation and the CFB equations. The Burgers equation has been thoroughly researched by many people over the years. This section provides a review of some of the previously established properties of the inviscid Burgers equation. Many of these will be used later on to establish new results about the CFB equations. This section will also list some of the previously established properties of the CFB equations, which are also crucial to the analysis found in the following sections.

2.1. Method of characteristics. The inviscid Burgers equation lends itself well to examination with the method of characteristics. From Whitham [13], the inviscid Burgers equation can be broken into two ODEs,

$$(2.1) \qquad u_t(\xi) = 0,$$

$$(2.2) \qquad \frac{\partial}{\partial t}\xi = u(\xi).$$

From this it is determined that along the characteristics

$$(2.3) \qquad \xi = x_0 + u_0(\xi)t,$$

$u(x)$ is constant. Thus characteristics travel at the speed equal to the value of u along those characteristics. This is true until characteristics cross, forming shocks. This is equivalent to seeing that the material derivative is zero [19].

2.2. Weak solutions and entropy conditions. Lax [14] addresses weak solutions and entropy solutions of conservation laws. From his work, much information can be gained about the solutions to the inviscid Burgers equation.

The first thing we learn is that any weak solution to the inviscid Burgers equation must satisfy the integral form of the conservation law, or

$$(2.4) \qquad \int_g^h u \, dx \bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \frac{-u^2}{2} \bigg|_g^h dt,$$

which must hold for any g and h and every time interval (t_1, t_2). A consequence of this are the Rankine–Hugoniot jump conditions. These dictate the speed at which any discontinuity can propagate. If s is the position of a shock, then

$$(2.5) \qquad \frac{d}{dt}s(t) = \frac{1}{2}\left[u(s^-) + u(s^+)\right].$$

Lax also establishes the existence and uniqueness of a weak solution to the inviscid Burgers equation which satisfies the so-called entropy condition

$$(2.6) \qquad u(s^-) > u(s^+),$$

where s is the location of a discontinuity. Thus the only discontinuities that are allowed to exist in this "entropy solution" are decreasing jumps.

Lax also states that for solutions satisfying the entropy condition, "every point can be connected by a backward drawn characteristic to a point on the initial line." Thus any value of the entropy solution, $u(x,t)$, can be traced back to the initial conditions. For discontinuous initial conditions, points traced back to the point of discontinuity can take on values between the left and right limits of the discontinuity, as is shown in subsection 5.1. For continuous initial conditions the entropy solution can be written as $u(x,t) = u_0(\phi(x,t))$, where $\phi(x,t)$ is an increasing function of x for any time, and $\phi(x,0) = x$.

Here we will define what will be referred to in this paper as a reparameterization of a function.

DEFINITION 2.1. *If $\phi(x,t)$ is an increasing function of x for any time, and $\phi(x,0) = x$, the function $f(\phi(x,t))$ will be called a reparameterization of the function f.*

It is clear that at any time t, a reparameterization of the function f cannot obtain values that are not obtained by f. It can, however, lack values that are found in f, as it was not dictated that ϕ be onto for all time. Looking back to the previous paragraph we can see that for continuous initial conditions, the entropy solution to the inviscid Burgers equation will be a reparameterization of the initial conditions.

2.3. Properties of the CFB equations. From previous work by our group [20, 21, 22, 12] the following theorem is established. It is presented here in its one-dimensional form.

THEOREM 2.2. *Let $g(x) \in W^{1,1}(\mathbb{R})$ and $u_0(x) \in C^1(\mathbb{R})$; then there exists a unique global solution $u(x,t) \in C^1(\mathbb{R}, \mathbb{R})$ to the following initial value problem:*

(2.7a) $$u_t + \bar{u} u_x = 0,$$

(2.7b) $$\bar{u} = g * u,$$

(2.7c) $$u(x,0) = u_0(x).$$

A sketch of the proof of the theorem is as follows. Examine the equations using the method of characteristics. Due to the nature of the equations, the infinity norm of u will be bounded for all time. By Young's inequality, $||\bar{u}_x||_\infty$ can thus be bounded for all time. The characteristics of the equations will not cross if their Jacobian remains nonzero. The rate of change of the Jacobian can be directly related to $\bar{u}_x$ by

(2.8) $$\frac{\partial}{\partial t} J = \bar{u}_x \, J.$$

Since $||\bar{u}_x||_\infty$ remains bounded, the Jacobian will remain nonzero, the characteristics will not cross, and a unique solution will exist for any finite time.

In the course of proving the theorem, it was established that the solution take the form $u(x,t) = u_0(\phi(x,t))$, where $\phi(x,t)$ is a continuous, invertible, and increasing function of x for any time, and $\phi(x,0) = x$. Thus the solution is a reparameterization of its initial conditions.

3. Weak solution. Regularizations of conservation laws do not necessarily have to converge to weak solutions of those conservation laws. Take, for example, the Korteweg–de Vries (KdV) equations,

(3.1) $$u_t + u \, u_x = -\epsilon u_{xxx}.$$

This regularizes the inviscid Burgers equation in the sense that solutions are now continuous; however, many oscillations form as $\epsilon \to 0$, requiring a weak limit for convergence [23, 24]. This limit is not a weak solution of the inviscid Burgers equation [25], and thus is definitely not the entropy solution.

Thus the first step to proving convergence to the entropy solution is to prove convergence to a weak solution. The following subsections prove this by showing that a subsequence of the solutions to the CFB equations must converge to a function in L^1_{loc}. It is then shown that this function is, in fact, a weak solution to the inviscid Burgers equation.

3.1. Convergence of solutions. In this subsection we show that the solutions of the CFB equations (u^α) converge to a function u. This subsection mirrors work done by Bhat and Fetecau [18]. We begin by claiming the following properties of the solutions u^α.

LEMMA 3.1. *The solutions to the initial value problem* (1.3) *have the following properties:*

$$\|u^\alpha(\cdot, t)\|_{L^\infty} = \|u^\alpha(\cdot, 0)\|_{L^\infty} = \|u_0\|_{L^\infty} = A_1, \tag{3.2}$$

$$TV(u^\alpha(\cdot, t)) = TV(u^\alpha(\cdot, 0)) = TV(u_0(\cdot)) = A_2, \tag{3.3}$$

$$\int_{\mathbb{R}} |u^\alpha(x, t) - u^\alpha(x, s)| \, dx \le A_3 |t - s|, \tag{3.4}$$

where A_1, A_2, *and* A_3 *are independent of* α, *and* $TV(f(\cdot))$ *can be defined for a smooth function* f *as*

$$TV(f(\cdot)) = \int_{\mathbb{R}} |f'(x)| \, dx. \tag{3.5}$$

Proof. Property (3.2) is verified by the existence proof in earlier papers [21, 22, 12] that $\|u^\alpha(\cdot, t)\|_{L^\infty} = \|u^\alpha(\cdot, 0)\|_{L^\infty}$.

To verify property (3.3), take the derivative of (1.3a), multiply by $sign(u_x)$, and integrate over the real line to obtain

$$\frac{\partial}{\partial t} \int |u_x| \, dx + \int sign(u_x)(\bar{u} u_x)_x \, dx = 0. \tag{3.6}$$

Break the second term into intervals where $sign(u_x)$ remains constant. u_x and $\bar{u}$ are continuous due to previous existence theorems, so at the locations that $sign(u_x)$ switches signs, the value of u_x will be 0. Thus the second term is zero and we obtain the result

$$\|u_x(\cdot, t)\|_{L^1} = \|u_x(\cdot, 0)\|_{L^1}, \tag{3.7}$$

and thus property (3.3) is established.

Property (3.4) can be proved by the following estimate:

$$\int_{\mathbb{R}} |u^\alpha(x, t) - u^\alpha(x, s)| \, dx \le \int_{\mathbb{R}} \int_s^t |u_t^\alpha| \, dt \, dx$$

$$= \int_{\mathbb{R}} \int_s^t |\bar{u}^\alpha u_x^\alpha| \, dt \, dx$$

$$= \int_s^t \int_{\mathbb{R}} |\bar{u}^\alpha u_x^\alpha| \, dx \, dt$$

$$\le \|\bar{u}^\alpha\|_{L^\infty} \int_s^t \|u_x^\alpha\|_{L^1} \, dt$$

$$\le A_1 A_2 |t - s|. \qquad \square$$

From Bressan [26] and Serre [27] we know that properties (3.2), (3.3), and (3.4) are enough to guarantee that a subsequence of u^α converges to a function u in L^1_{loc}. Furthermore, u shares the same infinity norm bound as that established in (3.2), and it shares the same total variation bound as that in (3.3).

3.2. Convergence to a weak solution. To begin we look at a specific subset of filters. The filters we examine are the functions whose Fourier transforms can be written as

$$\hat{g}(k) = \frac{1}{1 + \sum_{j=1}^n C_j k^{2j}} \qquad \text{with } n < \infty, \, C_j \ge 0, \, C_n \ne 0.$$

Noting that $\hat{g}\hat{u} = \hat{\bar{u}}$, we can see that

$$\hat{u} = \left(1 + \sum_{j=1}^{n} C_j k^{2j}\right) \hat{\bar{u}}$$

and

$$u = \left(1 + \sum_{j=0}^{n} (-1)^i C_j \frac{\partial^{2j}}{\partial x^{2j}}\right) \bar{u}.$$

We will refer to a filter of this form as satisfying condition A. This class of filters includes the Helmholtz filter, which has been of previous interest in turbulence modeling.

Clearly $g(x)$ and its derivatives up to $g^{(2n-2)}(x)$ are well defined and bounded as $\frac{(ik)^{2n-2}}{1+\sum_{j=1}^{n} C_j k^{2j}}$ is absolutely integrable.

If u and its derivative u_x are absolutely integrable, then for a g satisfying condition A, the convolution

$$\frac{\partial^j}{\partial x^j} \bar{u} = \frac{\partial^{j-1}}{\partial x^{j-1}} g^\alpha * u_x$$

is well defined. Furthermore, by Young's inequality,

$$\left\| \frac{\partial^j}{\partial x^j} \bar{u} \right\|_\infty \leq \left\| \frac{\partial^{j-1}}{\partial x^{j-1}} g^\alpha \right\|_\infty \|u_x\|_1 = \frac{1}{\alpha^j} \left\| g^{(j-1)} \right\|_\infty \|u_x\|_1.$$

Thus there exists a constant A_4 such that

$$\left\| \frac{\partial^j}{\partial x^j} \bar{u} \right\|_\infty < \frac{1}{\alpha^j} A_4 \qquad \text{for } j \leq 2n - 1.$$

These criteria are used in the following lemma.

LEMMA 3.2. *Let u^α be a sequence of functions that satisfy the following conditions:*

(3.8a) $u^\alpha, \bar{u}^\alpha < A_1,$

(3.8b) $\int |u_x^\alpha| \, dx, \int |\bar{u}_x^\alpha| \, dx < A_2,$

(3.8c) $\left\| \frac{\partial^j}{\partial x^j} \bar{u}^\alpha \right\|_\infty < \frac{1}{\alpha^j} A_4 \qquad \text{for } j \leq 2n - 1.$

Let $f \in C^\infty$ be compactly supported on $\mathbb{R}$. Then as $\alpha \to 0$, the quantity

(3.9) $\alpha^{2n} \int_{-\infty}^{\infty} \left(\frac{\partial^{2n}}{\partial x^{2n}} \bar{u}^\alpha \right) \bar{u}_x^\alpha f \, dx$

limits to 0.

Proof. For convenience the u^α shall be denoted u. Integrate (3.9) by parts to obtain

(3.10) $\alpha^{2n} \int \bar{u}^{(2n)} \bar{u}_x f \, dx = \alpha^{2n} \int \bar{u}^{(n)} \frac{\partial^n}{\partial x^n} (\bar{u}_x f) \, dx.$

Use the product rule to expand $\left(\frac{\partial^n}{\partial x^n} \bar{u}_x f\right)$:

$$(3.11) \qquad \alpha^{2n} \int \bar{u}^{(n)} \sum_{i=0}^{n} \binom{n}{i} \bar{u}^{(1+i)} f^{(n-i)} \, dx.$$

Take the absolute value, separate the last two terms of the binomial expansion, and apply the triangle inequality:

$$(3.12) \qquad \leq \left| \alpha^{2n} \int \bar{u}^{(n)} \bar{u}^{(n+1)} f \, dx \right|$$

$$(3.13) \qquad + \left| \alpha^{2n} n \int \bar{u}^{(n)} \bar{u}^{(n)} f^{(1)} \, dx \right|$$

$$(3.14) \qquad + \left| \alpha^{2n} \int \bar{u}^{(n)} \sum_{i=0}^{n-2} \binom{n}{i} \bar{u}^{(1+i)} f^{(n-i)} \, dx \right|.$$

Begin by bounding the third term,

$$(3.15)$$
$$\underbrace{\left| \alpha^{2n} \int \bar{u}^{(n)} \sum_{i=0}^{n-2} \binom{n}{i} \bar{u}^{(1+i)} f^{(n-i)} \, dx \right|}_{\text{3rd term}} \leq \alpha^{2n} \sum_{i=0}^{n-2} \binom{n}{i} ||\bar{u}^{(n)}||_{\infty} ||\bar{u}^{(1+i)}||_{\infty} ||f^{(n-i)}||_{1}.$$

By applying the bound on $||\bar{u}^{(i)}||_{\infty}$,

$$(3.16) \qquad \text{3rd term} \leq \sum_{i=0}^{n-2} \binom{n}{i} \alpha^{n-i-1} A_4 ||f^{(n-i)}||_{1},$$

which limits to 0 as $\alpha \to 0$.

Next, deal with the second term:

$$(3.17) \qquad \underbrace{\left| \alpha^{2n} n \int \bar{u}^{(n)} \bar{u}^{(n)} f^{(1)} \, dx \right|}_{\text{2nd term}}$$

$$(3.18) \qquad = \left| \alpha^{2n} n \int \bar{u}^{(1)} \frac{\partial^{n-1}}{\partial x^{n-1}} (\bar{u}^{(n)} f^{(1)}) \, dx \right|$$

$$(3.19) \qquad = \left| \alpha^{2n} n \int \bar{u}^{(1)} \sum_{i=0}^{n-1} \binom{n-1}{i} (\bar{u}^{(n+i)} f^{(n-i)}) \, dx \right|$$

$$(3.20) \qquad \leq \alpha^{2n} n \sum_{i=0}^{n-1} \binom{n-1}{i} ||\bar{u}^{(1)}||_{1} ||\bar{u}^{(n+i)}||_{\infty} ||f^{(n-i)}||_{\infty}.$$

Again, apply the bound on $||\bar{u}^{(i)}||_{\infty}$ to get

$$(3.21) \qquad \text{2nd term} \leq n \sum_{i=0}^{n-1} \alpha^{n-i} \binom{n-1}{i} A_4 ||\bar{u}^{(1)}||_{1} ||f^{(n-i)}||_{\infty}.$$

Since f and all its derivatives are bounded and $||\bar{u}^{(1)}||_{1} < A_2$, the second term also limits to zero.

Now we show the first term:

$$(3.22) \qquad \left| \alpha^{2n} \int \bar{u}^{(n)} \bar{u}^{(n+1)} f \, dx \right| = \left| \alpha^{2n} \int \frac{1}{2} \frac{\partial}{\partial x} (\bar{u}^{(n)})^2 f \, dx \right|$$

$$(3.23) \qquad = \left| \frac{\alpha^{2n}}{2} \int \bar{u}^{(n)} \bar{u}^{(n)} f^{(1)} \, dx \right|.$$

This differs from the second term only by a constant, so it must limit to 0 as $\alpha \to 0$.

Thus we obtain the result

$$(3.24) \qquad \lim_{\alpha \to 0} \alpha^{2n} \int_{-\infty}^{\infty} \left(\frac{\partial^{2n}}{\partial x^{2n}} \bar{u} \right) \bar{u}_x f \, dx = 0. \qquad \square$$

The last piece needed is taken from Duoandikoetxea [28]. The following lemma is a restatement of Duoandikoetxea's Theorem 2.1 from [28, page 25].

LEMMA 3.3. *Let g be an integrable function on $\mathbb{R}$ such that $\int g = 1$. Define $g^\alpha = \frac{1}{\alpha} g(\frac{x}{\alpha})$. Then*

$$\lim_{\alpha \to 0} ||g^\alpha * f - f||_p = 0$$

if $f \in L^p$, $1 \leq p < \infty$, and uniformly (i.e., when $p = \infty$) if $f \in C_0(\mathbb{R})$.

With Lemmas 3.3 and 3.2 we can now prove the following theorem regarding convergence to weak solutions.

THEOREM 3.4. *For any g satisfying condition A, the solutions u^α to the CFB equations converge to a weak solution of the inviscid Burgers equation.*

Proof. It was already shown that u^α converges to a function u. To show this is a weak solution of the inviscid Burgers equation, we need to prove that for any test function $f \in C^\infty$ that has compact support on $\mathbb{R} \times [0, T]$ that

$$(3.25) \qquad \int_0^T \int_{\mathbb{R}} u f_t + \frac{1}{2} u^2 f_x \, dx \, dt = 0.$$

Begin by rewriting (1.3a) as

$$(3.26) \qquad u_t^\alpha + \left(\frac{1}{2} (\bar{u}^\alpha)^2 \right)_x = (\bar{u}_x^\alpha - u_x^\alpha) \bar{u}^\alpha.$$

Multiply by the test function f and integrate over $\mathbb{R} \times [0, T]$,

$$(3.27) \qquad \int_0^T \int_{\mathbb{R}} u_t^\alpha f + \left(\frac{1}{2} (\bar{u}^\alpha)^2 \right)_x f \, dx \, dt = \int_0^T \int_{\mathbb{R}} (\bar{u}_x^\alpha - u_x^\alpha) \bar{u}^\alpha f \, dx \, dt.$$

Integrate by parts,

$$\int_0^T \int_{\mathbb{R}} u^\alpha f_t + \left(\frac{1}{2} (\bar{u}^\alpha)^2 \right) f_x \, dx \, dt = \int_0^T \int_{\mathbb{R}} (\bar{u}^\alpha - u^\alpha) \bar{u}^\alpha f_x \, dx \, dt$$

$$(3.28) \qquad \qquad \qquad \qquad + \int_0^T \int_{\mathbb{R}} (\bar{u}^\alpha - u^\alpha) \bar{u}_x^\alpha f \, dx \, dt.$$

Taking the limit as $\alpha \to 0$ of the left-hand side, we obtain

$$\int_0^T \int_{\mathbb{R}} u f_t + \left(\frac{1}{2} (u)^2 \right) f_x \, dx \, dt.$$

Clearly, if the right-hand side limits to zero, then we have that u is a weak solution to the Burgers equation.

Begin with the first term on the right-hand side of (3.28). The first term can be shown to limit to zero by noting that $||u^\alpha||_\infty$ has a uniform bound of A_1, and that since $f \in C^\infty$ with compact support, there exists an $F \in \mathbb{R}^+$ such that $||f||_\infty \leq F$ and $||f_x||_\infty \leq F$. Additionally let f be supported on the compact set Ω. This leads to the bound

$$(3.29) \qquad \int_0^T \int_\mathbb{R} (u^\alpha - \bar{u}^\alpha) u^\alpha f_x \, dx \, dt \leq F \, A_1 \, T ||u^\alpha - \bar{u}^\alpha||_{L^1(\Omega)}.$$

Take the limit of $||u^\alpha - \bar{u}^\alpha||_{L^1(\Omega)}$. Break apart the norm with the triangle inequality to get

$$\lim_{\alpha \to 0} ||u^\alpha - u^\alpha * g^\alpha|| \leq \lim_{\alpha \to 0} ||u^\alpha - u|| + ||u - u * g^\alpha|| + ||g^\alpha * (u - u^\alpha)||$$
$$\leq \lim_{\alpha \to 0} ||u^\alpha - u|| + ||u - u * g^\alpha|| + ||g^\alpha|| \, ||u - u^\alpha||,$$

where the norms are all $|| \cdot ||_{L^1(\Omega)}$. The first and third terms limit to zero as u^α converges to u in L^1_{loc}. The second term limits to zero by Lemma 3.3.

Now deal with the second term from (3.28). Since g satisfies condition A,

$$(\bar{u}^\alpha - u^\alpha) = \sum_{j=1}^n C_i \alpha^{2j} \frac{\partial^{2j}}{\partial x^{2j}} \bar{u}^\alpha,$$

the second term can be rewritten as

$$\sum_{j=1}^n C_i \int_0^T \int_\mathbb{R} \alpha^{2j} \frac{\partial^{2j}}{\partial x^{2j}} \bar{u}^\alpha \bar{u}_x^\alpha f \, dx \, dt.$$

By Lemma 3.2 every term in the sum limits to zero. Hence the sum limits to zero.

Therefore the limit as $\alpha \to 0$ of (3.28) becomes

$$(3.30) \qquad \int_0^T \int_\mathbb{R} u f_t + \frac{1}{2} u^2 f_x \, dx \, dt = 0,$$

proving u is a weak solution of the inviscid Burgers equation. $\qquad \square$

4. Convergence to the entropy solution. In this section we will first examine some of the properties of nonentropic solutions, that is, solutions that are a weak solution to the inviscid Burgers equation but do not satisfy the entropy condition. By examining these properties, it will be shown that the solutions to the CFB equation lack certain properties found in all nonentropic solutions. Thus it will be shown that the solutions to the CFB equations converge to the entropy solution of the inviscid Burgers equation.

This examination will be limited to a class of initial conditions. Specifically, we intend to examine initial conditions that are continuously differentiable and are bell shaped, i.e., have an interval where the functions are increasing, followed by an interval where the functions are decreasing. Functions that satisfy this condition will be referred to as satisfying condition B. It is for these functions as initial conditions that we will prove convergence to the entropy solution.

DEFINITION 4.1. *Let $u(x) \in C^1(\mathbb{R})$ and $u_x \geq 0$ over $(-\infty, p)$ and $u_x \leq 0$ over $(p, -\infty)$ for some p. Additionally let $u(x)$ have finite limits as $x \to \pm\infty$. Then $u(x)$ is said to have satisfied condition B.*

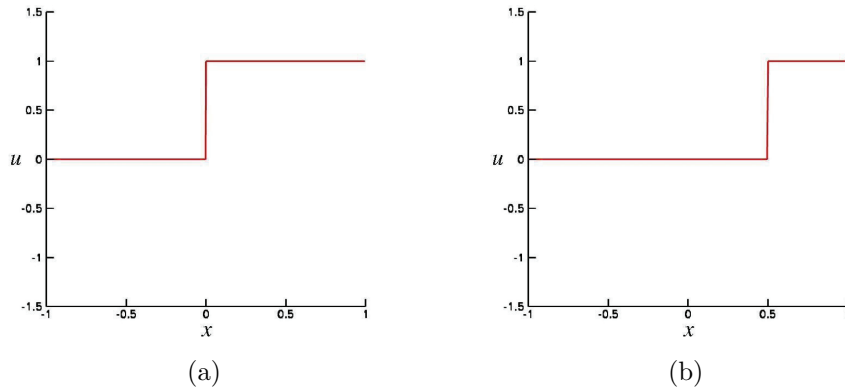(a)   (b)

FIG. 4.1. *Here a discontinuity is introduced in the initial conditions and remains. The shock must travel at the speed dictated by the Rankine–Hugoniot jump conditions to be a weak solution.*

4.1. Nonentropic weak solutions. There are three classic types of entropy violating weak solutions to the inviscid Burgers equation. This subsection shows examples of each type. The first starts with an increasing shock in the initial conditions, and then that shock remains, propagating at the speed dictated by the Rankine–Hugoniot jump conditions. An example of this is

$$u(x,t) = \begin{cases} 0 & \text{if } x < \frac{1}{2}t, \\ 1 & \text{if } \frac{1}{2}t \leq x, \end{cases}$$

taken from Lax [14] and illustrated in Figure 4.1.

The second case is when a shock already exists and then splits into multiple shocks, one of which is an entropy violating shock. All the shocks move with the speed dictated by the Rankine–Hugoniot conditions. For $a \geq 1$ the following is a weak solution to the Burgers equation. This example was taken from Oleinik [15] and is illustrated in Figure 4.2:

$$u(x,t) = \begin{cases} 1 & \text{if } x < \frac{1-a}{2}t, \\ -a & \text{if } \frac{1-a}{2}t \leq x < 0, \\ a & \text{if } 0 \leq x < \frac{a-1}{2}t, \\ -1 & \text{if } \frac{a-1}{2}t \leq x. \end{cases}$$

Another example is spontaneous shock formation with shocks forming out of a continuous interval. For $a > 0$ the following is a weak solution to the Burgers equation. This example was taken from Serre [27] and is illustrated in Figure 4.3:

$$u(x,t) = \begin{cases} 0 & \text{if } x < \frac{-a}{2}t, \\ -a & \text{if } \frac{-a}{2}t \leq x < 0, \\ a & \text{if } 0 \leq x < \frac{a}{2}t, \\ 0 & \text{if } \frac{a}{2}t \leq x. \end{cases}$$

In the next subsection it is shown that these three cases exemplify the only type of entropy violating behavior possible.
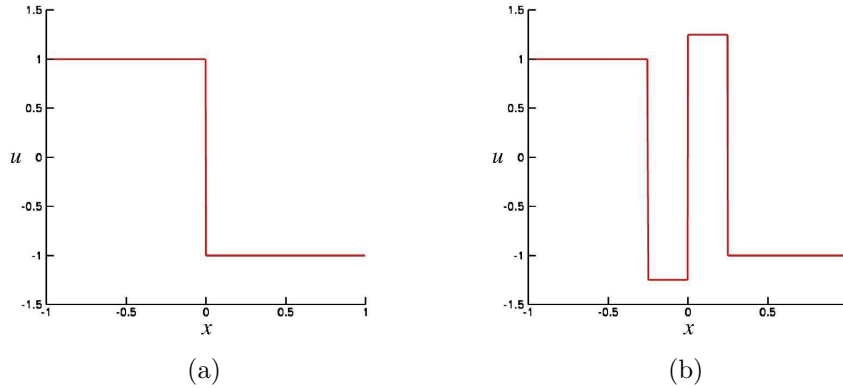
FIG. 4.2. *A shock can split into multiple shocks and still remain a weak solution.* (a) *The initial conditions.* (b) *The solution after the shock splitting has occurred.*
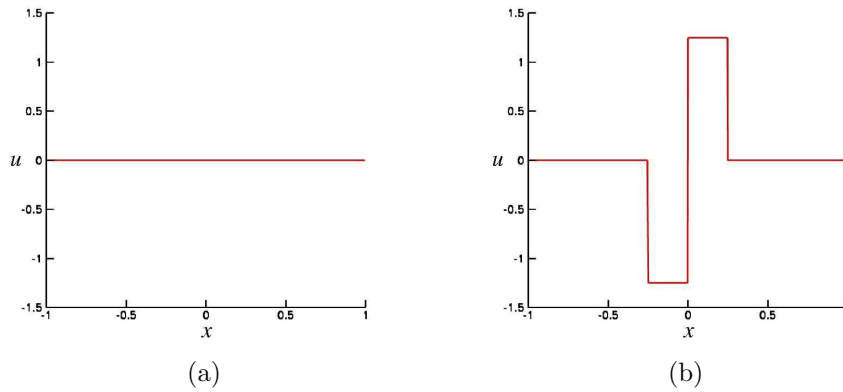

FIG. 4.3. *A shock can form from a continuous interval.* (a) *The initial conditions.* (b) *The solution after spontaneous shock formation has occurred.*

4.2. Decreasing slope along characteristics. By examining the inviscid Burgers equation, it is possible to see that a nonentropic solution cannot form through the steepening of the solution. With this information we can then limit the ways a nonentropic solution can form. Begin with the inviscid Burgers equation,

$$(4.1) \qquad u_t + u u_x = 0.$$

In section 2 it was seen that along the characteristics,

$$(4.2) \qquad \xi = x_0 + u_0(\xi)t,$$

the value of u remains constant. This is true until characteristics cross, at which point a shock is formed.

Here a similar approach is taken, but on the derivative of the inviscid Burgers equation. Differentiate the inviscid Burgers equation to get

$$(4.3) \qquad \frac{d}{dt}(u_x) + u\frac{d}{dx}(u_x) = -(u_x)^2.$$

Now if we examine this equation, we find that along the same characteristics $\xi = x_0 + u_0(\xi)t$ the quantity u_x is governed by

$$(4.4) \qquad \frac{d}{dt}u_x = -(u_x)^2.$$

Thus for piecewise differentiable solutions, u_x is always decreasing along characteristics and an increasing shock cannot form from the steepening of the solution.

Now consider a solution that begins with initial conditions satisfying condition B. That solution is a continuously differentiable solution to the inviscid Burgers equation and is thus an entropy solution. It will remain an entropy solution until an increasing jump is formed. An entropy solution for initial conditions satisfying condition B will be piecewise continuous, and thus from above will not steepen into an increasing shock. From this we conclude that an increasing shock can occur only if it exists in the initial conditions or must form instantaneously as it cannot form from the steepening of the solution. It can either form at existing points of discontinuity or form at points of continuity, which this paper refers to as shock splitting and spontaneous shock formation, respectively.

4.3. Entropy violating solutions are not reparameterizations of initial conditions. In section 2 it was established that the entropy solution of the inviscid Burgers equation is a reparameterization of initial conditions when the initial conditions are continuous. This subsection shows that a nonentropic solution cannot be both a weak solution and a reparameterization of initial conditions satisfying condition B.

We first begin be examining some consequences of being both a weak solution and a reparameterization of initial conditions satisfying condition B. Then we assume that there is a nonentropic solution that is both a weak solution and a reparameterization and show that this is a contradiction.

If a function is a reparameterization of initial conditions satisfying condition B, it is easy to see that the reparameterization will have one interval, where it is increasing, followed by an interval where it is decreasing. It is also clearly bounded. However, it need not be continuous. As a direct consequence of the monotone convergence theorem for sequences, every point on the reparameterization will have a well-defined left- and right-sided limit. Since the left- and right-sided limits are well defined, the only type of discontinuity allowed is a jump discontinuity. If a more rigorous explanation is desired, we refer the reader to section 5.7 in Davidson and Donsig [29].

Additionally any function satisfying condition B will have bounded variation. Thus any function that is a reparameterization will have variation bounded by the original function's variation. Thus if a solution is a reparameterization of initial conditions satisfying condition B, then it is of bounded variation.

From Theorem 1.8.1 on pages 21 and 52 in Dafermos [30] we know that a function u that is of class BV_{loc} and is a weak solution will satisfy the Rankine–Hugoniot jump conditions at every jump discontinuity. This means that if χ is the location of a discontinuity, then

$$(4.5) \qquad \frac{d}{dt}\chi = \frac{u(\chi^-, t) + u(\chi^+, t)}{2}.$$

Thus if the solution is a weak solution and a reparameterization of initial conditions satisfying condition B, all of its discontinuities must be jump discontinuities satisfying the Rankine–Hugoniot jump conditions.
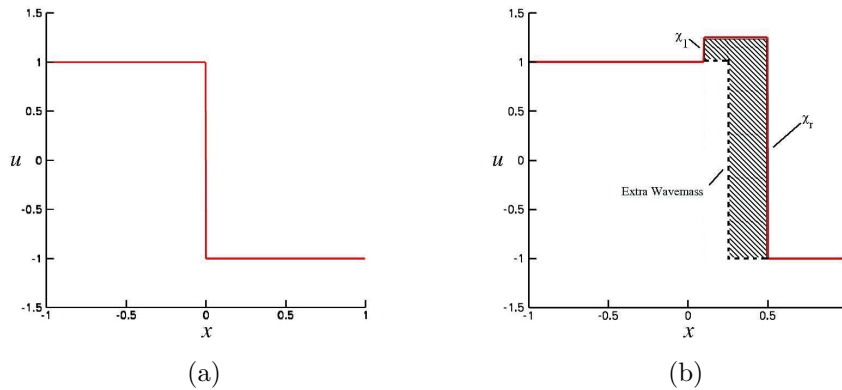
FIG. 4.4. *Wavemass is created in shock splitting or spontaneous shock formation if the middle area is greater than the outer area's values at the shocks.* (a) *The conditions of $u(x,t)$ before shock splitting.* (b) *The extra wavemass created when a shock splits and the middle area has higher values than its surroundings. Here χ_l and χ_r are the left- and rightmost shocks as used in Lemma 4.2. The extra wavemass is indicated.*

To show that a function is not a reparameterization of initial conditions satisfying condition B, it is sufficient to find three points $x_1 < x_2 < x_3$ such that $u(x_1) > u(x_2)$ and $u(x_2) < u(x_3)$. Essentially a function satisfying condition B is bell shaped, and finding these three points finds an upsidedown bell, which cannot happen in a reparameterization. This is precisely the method used to show that a nonentropic solution cannot be a reparameterization of the initial conditions.

Since we are considering only initial conditions satisfying condition B, we are beginning only with continuous initial conditions. Thus from subsection 4.2 the only possibility of having a nonentropic solution is through either spontaneous shock formation or shock splitting. It will be shown that if either of these occurs, then the nonentropic solution fails to be a reparameterization of the initial conditions.

Lemma 4.2 following is used later on when dealing with spontaneous shock formation and shock splitting. Because a nonentropic solution must still be a weak solution, spontaneous shock formation and shock splitting must behave in certain ways. The inviscid Burgers equation can be considered as a conservation law of wavemass, $\int u$. Lemma 4.2 uses this fact to place restrictions on how spontaneous shock formation and shock splitting can occur.

Lemma 4.2 addresses the area between the leftmost and the rightmost shock, when spontaneous shock formation or shock splitting occurs. Essentially it says that if the area between the shocks has a higher value than the value on the outside of the shocks, then wavemass has been created, and it is no longer a weak solution to the inviscid Burgers equation. Figure 4.4 shows an illustration of this.

The lemma proves that if the area between the leftmost and rightmost shock has values greater than those on its borders, then $u(x,t)$ cannot be a weak solution of the inviscid Burgers equation and a reparameterization of initial conditions. This is a proof by contradiction, so we assume that $u(x,t)$ is a weak solution and a reparameterization of initial conditions satisfying condition B, which places several constraints on $u(x,t)$. Such a weak solution to the inviscid Burgers equation must satisfy the Rankine–Hugoniot jump conditions (4.5).

Additionally, weak solutions must satisfy the integral form of the conservation

law,

$$(4.6) \qquad \int_g^h u \, dx \Bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \frac{-u^2}{2} \Bigg|_g^h dt,$$

or if g and h are moving boundaries, then

$$(4.7) \quad \int_{g(t)}^{h(t)} u \, dx \Bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \frac{-u^2}{2} \Bigg|_{g(t)}^{h(t)} + \left(\frac{d}{dt}h(t)\right) u(h(t),t) - \left(\frac{d}{dt}g(t)\right) u(g(t),t) \, dt.$$

The second definition, equation (4.7), is used in the following lemma.

LEMMA 4.2. *Assume that $u(x,t)$ takes the form*

$$(4.8) \qquad u(x,t) = \begin{cases} a(x,t) & \text{if } x < \chi_l(t), \\ b(x,t) & \text{if } \chi_l(t) \le x < \chi_r(t), \\ c(x,t) & \text{if } \chi_r(t) \le x, \end{cases}$$

where $\chi_l(t)$ and $\chi_r(t)$ are locations of discontinuities and $\chi_l(t_1) = \chi_r(t_1) = x^*$. At time t_1 let $a(x^{*-}, t_1) \ge c(x^{*+}, t_1)$. If for some period of time after t_1 and all $x \in (\chi_l, \chi_r)$, $b(x,t) > a(\chi_l(t)^-, t)$ and $b(x,t) > c(\chi_r(t)^+, t)$, then $u(x,t)$ cannot be a weak solution of the inviscid Burgers equation.

Proof. Begin by assuming that $u(x,t)$ is a weak solution of the inviscid Burgers equation and thus must satisfy (4.7) for any $g(t)$ and $h(t)$. We will start by considering the left-hand side of (4.7) with selected moving boundaries and show that it is strictly greater than the right-hand side, proving that $u(x,t)$ cannot be a weak solution by contradiction. The moving boundaries will be defined by the positions of the leftmost and rightmost shocks.

With the moving boundaries established, begin with the left-hand side of (4.7) for the given boundaries. By putting a bound on the integrand, we transform the spatial integral into a temporal integral:

$$(4.9) \qquad \underbrace{\int_{\chi_l(t)}^{\chi_r(t)} u \, dx \Bigg|_{t_1}^{t_2}}_{\text{LHS}} \ge \min_{x \in (\chi_l \chi_r)} b(x, t_2) \, (\chi_r - \chi_l)$$

$$(4.10) \qquad = \min_{x \in (\chi_l \chi_r)} b(x, t_2) \int_{t_1}^{t_2} \frac{\partial}{\partial t}\chi_r - \frac{\partial}{\partial t}\chi_l \, dt.$$

Now manipulate the equation to begin resembling the right-hand side of (4.7):

$$\begin{aligned}
\text{LHS} &\ge \int_{t_1}^{t_2} \min_{x \in (\chi_l \chi_r)} b(x, t_2) \left(\frac{\partial}{\partial t}\chi_r - \frac{\partial}{\partial t}\chi_l\right) dt \\
&= \int_{t_1}^{t_2} \left(\min_{x \in (\chi_l \chi_r)} b(x, t_2) - c(\chi_r, t)\right)\left(\frac{\partial}{\partial t}\chi_r - \frac{\partial}{\partial t}\chi_l\right) \\
&\quad + (a(\chi_l, t) - c(\chi_r, t))\frac{\partial}{\partial t}\chi_l \\
&\quad + c(\chi_r, t)\frac{\partial}{\partial t}\chi_r - a(\chi_l, t)\frac{\partial}{\partial t}\chi_l \, dt.
\end{aligned}$$

Use (4.15) to substitute in the speed of χ_l dictated by the Rankine–Hugoniot jump conditions:

$$\text{LHS} \geq \int_{t_1}^{t_2} \left(\min_{x \in (\chi_l \chi_r)} b(x, t_2) - c(\chi_r, t) \right) \left(\frac{\partial}{\partial t} \chi_r - \frac{\partial}{\partial t} \chi_l \right)$$
$$+ (a(\chi_l, t) - c(\chi_r, t)) \left(\frac{b(\chi_l, t) + a(\chi_l, t)}{2} \right)$$
$$+ c(\chi_r, t) \frac{\partial}{\partial t} \chi_r - a(\chi_l, t) \frac{\partial}{\partial t} \chi_l \ dt$$
$$= \int_{t_1}^{t_2} \underbrace{\left(\min_{x \in (\chi_l \chi_r)} b(x, t_2) - c(\chi_r, t) \right) \left(\frac{\partial}{\partial t} \chi_r - \frac{\partial}{\partial t} \chi_l \right)}_{\text{L}}$$
$$+ \underbrace{(a(\chi_l, t) - c(\chi_r, t)) \left(\frac{b(\chi_l, t) - c(\chi_r, t)}{2} \right)}_{\text{M}}$$
$$+ (a(\chi_l, t) - c(\chi_r, t)) \left(\frac{c(\chi_r, t) + a(\chi_l, t)}{2} \right) + c(\chi_r, t) \frac{\partial}{\partial t} \chi_r - a(\chi_l, t) \frac{\partial}{\partial t} \chi_l \ dt.$$

Consider term L. The value of $b(x, t)$ for all x and some period of time after t_1 was designated to be higher than $c(\chi_r, t)$. Additionally, for at least a short period of time, $\frac{\partial}{\partial t} \chi_r > \frac{\partial}{\partial t} \chi_l$; otherwise the interval (χ_l, χ_r) cannot have a nonzero measure. Thus for values t_2 close to t_1, term L is strictly positive.

Now consider term M. Again the value of $b(x, t)$ for all x and some period of time after t_1 was designated to be higher than $c(\chi_r, t)$. It was designated that at time t_1, $a(x^{*-}, t_1) \geq c(x^{*+}, t_1)$. If $a(x^{*-}, t_1) > c(x^{*+}, t_1)$, then for values t_2 close to t_1 term M is strictly positive. If $a(x^{*-}, t_1) = c(x^{*+}, t_1)$, then by choosing t_2 close to t_1, term M can be made arbitrarily small.

As t_2 approaches t_1, term L is approaching a strictly positive number, and term M is approaching a nonnegative number. Thus it is possible to choose a t_2 where $\int_{t_1}^{t_2} \text{L} + \text{M} \ dt > 0$. Using this we see that

$$(4.11) \qquad \text{LHS} > \int_{t_1}^{t_2} \frac{a(\chi_l, t)^2 - c(\chi_r, t)^2}{2} + c(\chi_r, t) \frac{\partial}{\partial t} \chi_r - a(\chi_l, t) \frac{\partial}{\partial t} \chi_l \ dt.$$

The right-hand side of (4.11) is the right-hand side of (4.7) with our chosen boundaries. Since with our moving boundaries the left-hand side of (4.7) is strictly greater than the right-hand side, $u(x, t)$ cannot be a weak solution. $\square$

This result is now used to show that if there is spontaneous shock formation or shock splitting, then $u(x, t)$ cannot be both a weak solution and a reparameterization of initial conditions.

4.3.1. Spontaneous shock formation. Assume that $u(x, t)$ is the entropy solution to the inviscid Burgers equation up to time t_1 where an increasing shock is formed spontaneously at point x^*. For such a discontinuity to form at least one other discontinuity must form in response. Thus we say after time t_1,

$$(4.12) \qquad u(x, t) = \begin{cases} a(x, t) & \text{if } x < \chi_l(t), \\ b(x, t) & \text{if } \chi_l(t) \leq x < \chi_r(t), \\ c(x, t) & \text{if } \chi_r(t) \leq x, \end{cases}$$
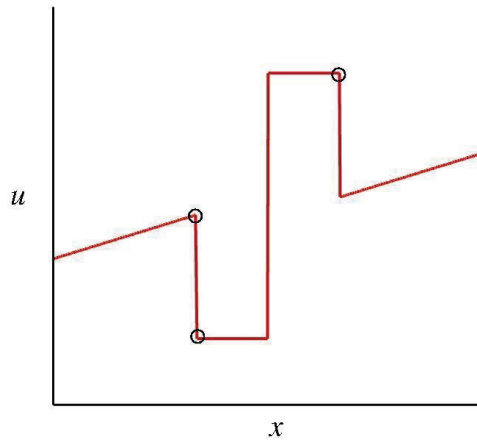
FIG. 4.5. *With two decreasing shocks, $u(x,t)$ cannot be a reparameterization. The circles represent the points chosen to prove that $u(x,t)$ cannot be a reparameterization of initial conditions.*

where $a(x,t)$, $b(x,t)$, and $c(x,t)$ are weak solutions to the inviscid Burgers equation, and $\chi_l(t)$ and $\chi_r(t)$ give the position of the leftmost and rightmost discontinuities formed during the shock splitting. There may be more than two shocks formed, as seen in subsection 4.1, but we need to examine just the leftmost and rightmost.

For $u(x,t)$ to be a weak solution we note that several things must be true. The speed of $\chi_l(t)$ and $\chi_r(t)$ is dictated by the Rankine–Hugoniot jump conditions to be

$$(4.13) \qquad \frac{d}{dt}\chi_l = \frac{a(\chi_l^-) + b(\chi_l^+)}{2}, \qquad \frac{d}{dt}\chi_r = \frac{b(\chi_r^-) + c(\chi_r^+)}{2}.$$

For there to be spontaneous shock forming, there must be some interval (t_1, t_2), where $\frac{d}{dt}\chi_r > \frac{d}{dt}\chi_l$. Thus for some interval (t_1, t_2), if $a(\chi_l^-) \geq c(\chi_r^+)$, then $b(\chi_l(t)^+, t) < b(\chi_r(t)^-, t)$. Assume that $t \in (t_1, t_2)$ for the remainder of the subsection.

The shocks located at $\chi_l(t)$ and $\chi_r(t)$ must be either increasing or decreasing shocks. We will examine each of the possibilities and show that each leads to $u(x,t)$ not being a reparameterization of the initial conditions.

Case 1. Assume that the shock at $\chi_l(t)$ is a decreasing shock and the shock at $\chi_r(t)$ is a decreasing shock. Then $a(\chi_l(t)^-, t) > b(\chi_l(t)^+, t)$ and $b(\chi_r(t)^-, t) > c(\chi_r(t)^+, t)$. If $b(\chi_l(t)^+, t) \geq b(\chi_r(t)^-, t)$, then by the transitive property $a(\chi_l(t)^-, t) > c(\chi_r(t)^+, t)$ and this violates the Rankine–Hugoniot condition, as was mentioned above, and $u(x,t)$ is not a weak solution. If $b(\chi_l(t)^+, t) < b(\chi_r(t)^-, t)$, then $a(\chi_l(t)^-, t) > b(\chi_l(t)^+, t) < b(\chi_r(t)^-, t)$ shows $u(x,t)$ is not a reparameterization of initial conditions. See Figure 4.5.

Case 2. Assume that the shock at $\chi_l(t)$ is a decreasing shock and the shock at $\chi_r(t)$ is an increasing shock. Then $a(\chi_l(t)^-, t) > b(\chi_l(t)^+, t)$ and $b(\chi_r(t)^-, t) < c(\chi_r(t)^+, t)$. Let $b_2 = \min(b(\chi_l(t)^+, t), b(\chi_r(t)^-, t))$; then $a(\chi_l(t)^-, t) > b_2 < c(\chi_r(t)^+, t)$ shows $u(x,t)$ is not a reparameterization of initial conditions. See Figure 4.6.

Case 3. Assume that the shock at $\chi_l(t)$ is an increasing shock and that the shock at $\chi_r(t)$ is an increasing shock. Since $a(\chi_l(t_1)^-, t_1) = c(\chi_r(t_1)^-, t_1)$, for at least a short period of time after t_1, the left value of $b(x,t)$ will be greater than $a(\chi_l(t), t)$ and $c(\chi_l(t), t)$, and the right value of $b(x,t)$ will be less than $a(\chi_l(t), t)$ and
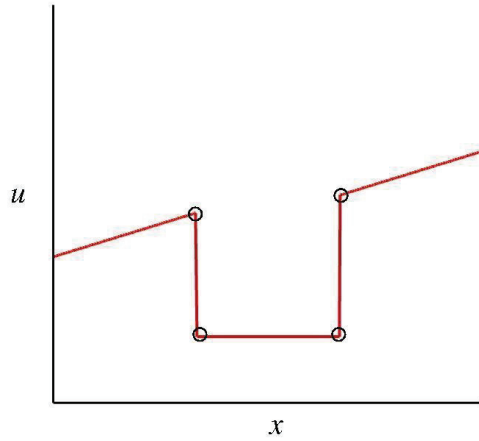
FIG. 4.6. *With a decreasing and an increasing shock, $u(x,t)$ cannot be a reparameterization. The circles represent the points chosen to prove that $u(x,t)$ cannot be a reparameterization of initial conditions.*
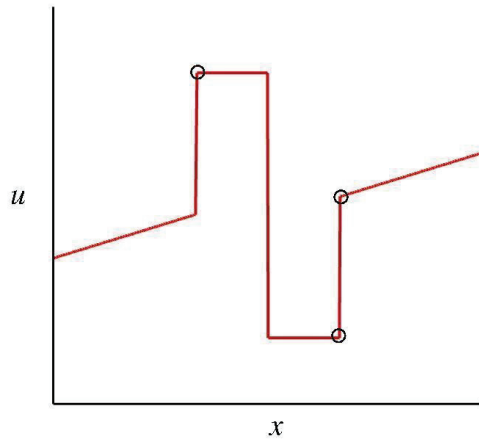

FIG. 4.7. *With two increasing shocks, $u(x,t)$ cannot be a reparameterization. The circles represent the points chosen to prove that $u(x,t)$ cannot be a reparameterization of initial conditions.*

$c(\chi_l(t), t)$. By choosing the points $b(\chi_l(t)^+, t) > b(\chi_r(t)^-, t) < c(\chi_r(t)^+, t)$, $u(x,t)$ is not a reparameterization of the initial conditions. See Figure 4.7.

Case 4. Assume that the shock at $\chi_l(t)$ is an increasing shock and that the shock at $\chi_r(t)$ is a decreasing shock. This case will be divided into two subcases. The first is that for all $x \in (\chi_l(t), \chi_r(t))$, $b(x,t) > a(\chi_l(t)^-, t)$ and $b(x,t) > c(\chi_r(t)^+, t)$. If this is the case, then $u(x,t)$ is proven to not be a weak solution by Lemma 4.2.

The second case is that there exists an $x_1 \in (\chi_l(t), \chi_r(t))$ such that $b(x_1, t) \leq a(\chi_l(t)^-, t)$ or $b(x_1, t) \leq c(\chi_r(t)^-, t)$. Since $\chi_l(t)$ is an increasing shock and $\chi_r(t)$ is a decreasing shock, and $a(\chi_l(t_1)^-, t_1) = c(\chi_r(t_1)^-, t_1)$, for at least a short period of time after t_1, the left and right values of $b(x,t)$ will be greater than $a(\chi_l(t), t)$ and $c(\chi_l(t), t)$. Thus if $b(x_1, t) \leq a(\chi_l(t)^-, t)$ or $b(x_1, t) \leq c(\chi_r(t)^-, t)$, the points
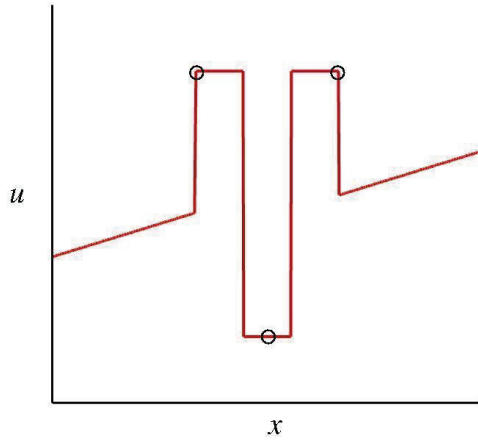
FIG. 4.8. *With a decreasing and an increasing shock, $u(x,t)$ cannot be a reparameterization. The circles represent the points chosen to prove that $u(x,t)$ cannot be a reparameterization of initial conditions.*

$b(\chi_l(t)^+, t) > b(x_1, t) < b(\chi_r(t)^+, t)$ show that $u(x,t)$ is not a reparameterization of initial conditions. See Figure 4.8.

Thus if $u(x,t)$ is a weak solution of the inviscid Burgers equation and a reparameterization of initial conditions, it cannot engage in spontaneous shock formation.

4.3.2. Shock splitting. Assume that $u(x,t)$ is the entropy solution to the inviscid Burgers equation up to time t_1, where an existing decreasing shock splits into two or more at point x^*. Thus we say after time t_1,

$$(4.14) \qquad u(x,t) = \begin{cases} a(x,t) & \text{if } x < \chi_l(t), \\ b(x,t) & \text{if } \chi_l(t) \le x < \chi_r(t), \\ c(x,t) & \text{if } \chi_r(t) \le x, \end{cases}$$

where $a(x,t)$, $b(x,t)$, and $c(x,t)$ are weak solutions to the inviscid Burgers equation and $\chi_l(t)$ and $\chi_r(t)$ give the position of the leftmost and rightmost discontinuities formed during the shock splitting. As there is assumed to be an already existing decreasing shock at time t_1, $a(\chi_l(t_1)^-, t_1) > c(\chi_r(t_1)^+, t_1)$. There may be more than two shocks formed, as seen in subsection 4.1, but we just need to examine the leftmost and rightmost.

For $u(x,t)$ to be a weak solution we note that several things must be true. The speed of $\chi_l(t)$ and $\chi_r(t)$ are dictated by the Rankine–Hugoniot jump conditions to be

$$(4.15) \qquad \frac{d}{dt}\chi_l = \frac{a(\chi_l^-) + b(\chi_l^+)}{2}, \qquad \frac{d}{dt}\chi_r = \frac{b(\chi_r^-) + c(\chi_r^+)}{2}.$$

For there to be shock splitting, there must be some interval (t_1, t_2), where $\frac{d}{dt}\chi_r > \frac{d}{dt}\chi_l$. Thus for some interval (t_1, t_2), $a(\chi_l^-) > c(\chi_r^+)$, and thus $b(\chi_l(t)^+, t) < b(\chi_r(t)^-, t)$. Assume that $t \in (t_1, t_2)$ for the remainder of the subsection.

The shocks located at $\chi_l(t)$ and $\chi_r(t)$ must be either increasing or decreasing shocks. We will examine each of the possibilities and show that each leads to $u(x,t)$ not being a reparameterization of the initial conditions.
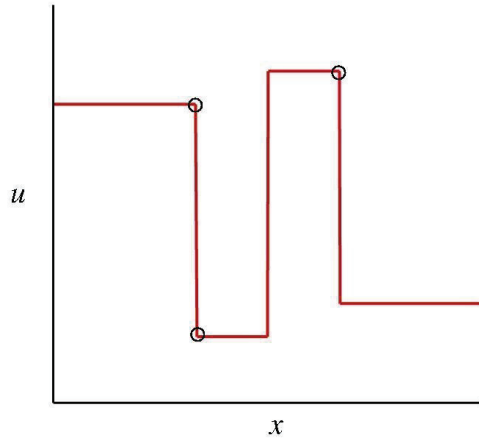
FIG. 4.9. *With two decreasing shocks, $u(x,t)$ cannot be a reparameterization. The circles represent the points chosen to prove that $u(x,t)$ cannot be a reparameterization of initial conditions.*
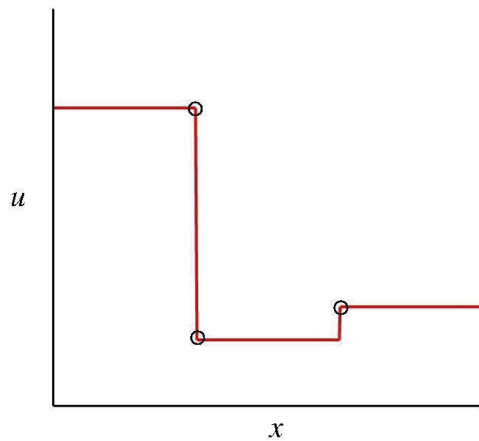

FIG. 4.10. *With a decreasing and an increasing shock, $u(x,t)$ cannot be a reparameterization. The circles represent the points chosen to prove that $u(x,t)$ cannot be a reparameterization of initial conditions.*

Case 1. Assume that the shock at $\chi_l(t)$ is a decreasing shock and the shock at $\chi_r(t)$ is a decreasing shock. Then $a(\chi_l(t)^-, t) > b(\chi_l(t)^+, t)$ and $b(\chi_r(t)^-, t) > c(\chi_r(t)^+, t)$. We know that $b(\chi_l(t)^+, t) < b(\chi_r(t)^-, t)$, and thus $a(\chi_l(t)^-, t) > b(\chi_l(t)^+, t) < b(\chi_r(t)^-, t)$, shows $u(x,t)$ is not a reparameterization of initial conditions. See Figure 4.9.

Case 2. Assume that the shock at $\chi_l(t)$ is a decreasing shock and the shock at $\chi_r(t)$ is an increasing shock. Then $a(\chi_l(t)^-, t) > b(\chi_l(t)^+, t)$ and $c(\chi_r(t)^+, t) > b(\chi_r(t)^-, t) > b(\chi_l(t)^+, t)$. Thus the points $a(\chi_l(t)^-, t) > b(\chi_l(t)^+, t) < c(\chi_r(t)^+, t)$ shows $u(x,t)$ is not a reparameterization of initial conditions. See Figure 4.10.

Case 3. Assume that the shock at $\chi_l(t)$ is an increasing shock and that the shock at $\chi_r(t)$ is an increasing shock. Then $a(\chi_l(t)^-, t) < b(\chi_l(t)^+, t)$ and $c(\chi_r(t)^+, t) >$
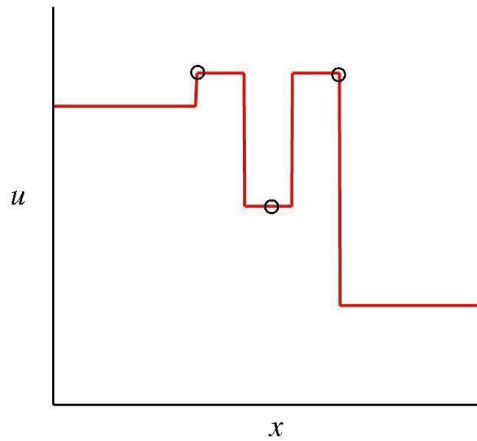
FIG. 4.11. *With a decreasing and an increasing shock, $u(x,t)$ cannot be a reparameterization. The circles represent the points chosen to prove that $u(x,t)$ cannot be a reparameterization of initial conditions.*

$b(\chi_r(t)^-, t)$. From the Rankine–Hugoniot jump conditions, for the interval (t_1, t_2), $a(\chi_l^-) > c(\chi_r^+)$, and thus $b(\chi_l(t)^+, t) < b(\chi_r(t)^-, t)$. This is a contradiction, so $u(x,t)$ is not a weak solution.

Case 4. Assume that the shock at $\chi_l(t)$ is an increasing shock and that the shock at $\chi_r(t)$ is a decreasing shock. This case will be divided into two subcases. The first is that for all $x \in (\chi_l(t), \chi_r(t))$, $b(x,t) > a(\chi_l(t)^-, t)$. If this is the case, then $u(x,t)$ is proven to not be a weak solution by Lemma 4.2.

The second case is that there exists an $x_1 \in (\chi_l(t), \chi_r(t))$ such that $b(x_1, t) \leq a(\chi_l(t)^-, t)$. Since $\chi_l(t)$ is an increasing shock, $b(\chi_l(t)^+, t) > a(\chi_l(t), t)$ and thus $b(\chi_l(t)^+, t) > b(x_1, t)$. Since $b(\chi_l(t)^+, t) < b(\chi_r(t)^-, t)$, the points $b(\chi_l(t)^+, t) > b(x_1, t) < b(\chi_r(t)^-, t)$ show that $u(x,t)$ is not a reparameterization of initial conditions. See Figure 4.11.

Thus if $u(x,t)$ is a weak solution of the inviscid Burgers equation and a reparameterization of initial conditions, it cannot engage in spontaneous shock formation or shock splitting.

4.3.3. Entropy violating solutions are not reparameterizations of initial conditions. From the previous subsections the following lemma can be established.

LEMMA 4.3. *Let $u(x,t)$ be a weak solution of the inviscid Burgers equation where the initial conditions satisfy condition* B. *If $u(x,t)$ is a reparameterization of initial conditions, then it is the entropy solution.*

Proof. Clearly this follows from the results in subsections 4.2, 4.3.1, and 4.3.2. ☐

4.4. Convergence to the entropy solution. Based on Lemma 4.3, the following theorem regarding the CFB equations converging to the entropy solution can be established.

THEOREM 4.4. *The solutions u^α of the CFB equations converge to the entropy solution of the inviscid Burgers equation for initial conditions satisfying condition* B.

Proof. It has already been established that u^α converges to a weak solution of the inviscid Burgers equation, u, in L^1_{loc}. In the existence and uniqueness proof, it

was established that u^α is a reparameterization of initial conditions. Clearly if every u^α is a reparameterization, then its limit u will also be a reparameterization of initial conditions. Since u is a reparameterization of initial conditions and a weak solution to the inviscid Burgers equation, by Lemma 4.3, u must be the entropy solution. ☐

We have established that the solutions of the CFB equations converge to the entropy solution of the inviscid Burgers equation for initial conditions satisfying condition B. The following section deals with how to regain the entropy solution for discontinuous initial conditions and why we believe that this result holds true for more general cases.

5. Extension into discontinuous initial conditions. Section 4 proves that the CFB equations will converge to the entropy solution for a specific set of initial conditions. This section explains the intuitive reasoning of why it is suspected that the CFB equations will converge to the entropy solution for any continuous initial conditions and why it will not for discontinuous initial conditions. It then shows how the equations can be changed slightly to incorporate discontinuous initial conditions. We begin with a commonly examined problem for the inviscid Burgers equation.

5.1. Example of entropic and nonentropic behavior for the inviscid Burgers equation. Consider the initial conditions

$$(5.1) \qquad u_0(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

The method of characteristics does not provide the value of u in the wedge $0 < x < t$, as seen in Figure 5.1. The entropy solution fills this wedge with the function $u(x,t) = \frac{x}{t}$ with characteristics fanning out from the original discontinuity, as seen in Figure 5.2(a). This creates a rarefaction wave and eliminates the discontinuity after time $t = 0$. A nonentropic solution will allow the discontinuity to continue to exist. It will fill the wedge with new characteristics which continuously originate from the discontinuity as time progresses, as seen in Figure 5.2(b). Thus the nonentropic solution creates new "information" as time progresses.

This problem embodies the essential behavior of entropic and nonentropic solutions and provides the basis for our reasoning in the following subsections.

5.2. Convergence to entropy solution for all continuous initial conditions. In section 4, it was proven that for initial conditions satisfying condition B, the solutions to the CFB equations converge to the entropy solution. It is the conjecture of this paper that the solutions to the CFB equations converge to the entropy solution for all continuous initial conditions. As mentioned above, a nonentropic solution will create new characteristics, or new information, as time progresses. The solutions to the CFB equation do not. The existence and uniqueness theorem proven in our previous paper [12], and restated here as Theorem 2.2, established that the solution takes the form $u(x,t) = u_0(\phi(x,t))$, where $\phi(x,t)$ is an increasing function of x for any time, and $\phi(x,0) = x$. This shows that no new information is being created in the CFB equations. Since the solutions to the CFB equations are converging to a weak solution to the inviscid Burgers equation and no new information is being created, it is reasonable to expect the solutions to converge to the entropy solution.

5.3. Nonconvergence for discontinuous initial conditions. Consider initial conditions that have an increasing discontinuity in them. The entropy solution to the inviscid Burgers equation creates a rarefaction wave from the discontinuity which takes
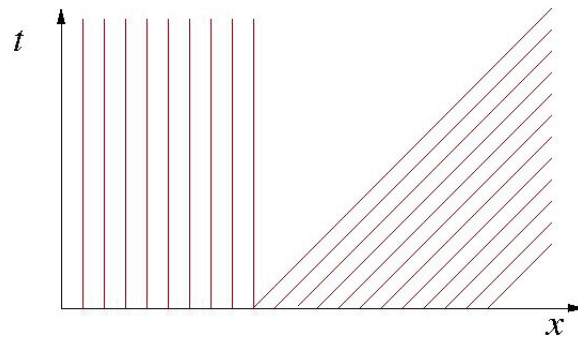
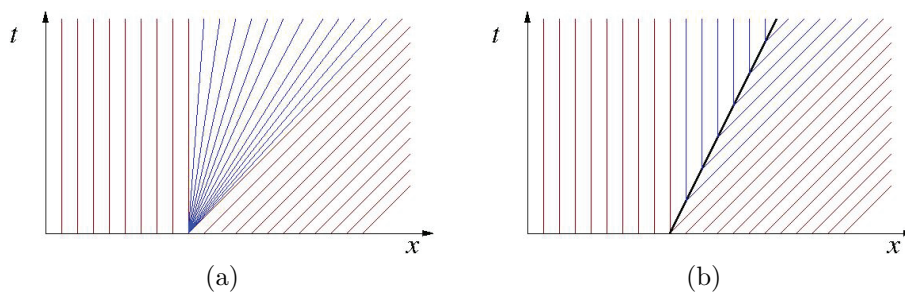FIG. 5.1. *The area $0 < x < t$ is not filled by characteristics.*

(a)   (b)

FIG. 5.2. *This is how entropic and nonentropic solutions fill the empty wedges.* (a) *The entropic solution fills the wedge with a rarefaction wave.* (b) *The nonentropic solution maintains the discontinuity by creating new characteristics.*

on all the values spanned by the discontinuity. No new characteristics are formed, as all originate from the discontinuity at time $t = 0$, but $u(x,t)$ now has values that did not originally exist in u_0. As shown in the existence and uniqueness theorem for the CFB equations, the solutions to the CFB equations must have only the values found in u_0. Thus for initial conditions containing an increasing discontinuity, the CFB equations will not converge to the entropy solution. An example of this can be found in our previous paper [12], in section 6, where a traveling wave solution to the CFB equations can be seen to converge to a nonentropic weak solution. For this reason, we eliminate discontinuous initial conditions from the admissible class of initial conditions.

5.4. Conjecture. Based on the reasoning in the previous two subsections, we present the following conjecture.

CONJECTURE 5.1. *The solutions u^α of the CFB equations converge to the entropy solution of the inviscid Burgers equation for continuous initial conditions as $\alpha \to 0$.*

Assuming this conjecture is true, there is still the matter of discontinuous initial conditions. The following subsection creates a new system that, if the conjecture is true, will converge to the entropy solution for all bounded initial conditions.

5.5. Regaining discontinuous initial conditions. In regarding discontinuous initial conditions, begin by assuming that for all C^1 initial conditions the solutions to the CFB equations converge to the entropy solution. Then if the C^1 initial conditions limit to the discontinuous initial conditions in L^1_{loc}, at the same time as $\alpha \to 0$, then the solutions will converge to the entropy solution for the discontinuous initial conditions. To prove this we use a theorem proven by Oleinik [15].

THEOREM 5.2. *Let $u^n(x,t)$ be the entropy solution for the inviscid Burgers equation with initial conditions $u^n(x,0) = u_0^n(x)$ and $u_0^n(x) \leq m$ for all n. Let*

$$\int_{-\infty}^{\infty} f(x) \left[u_0^n(x) - u_0(x) \right] dx \to 0$$

for $n \to \infty$ for any compactly supported continuous function $f(x)$. Then the sequence $u^n(x,t)$ converges for $n \to \infty$ to the entropy solution $u(x,t)$ in L^1_{loc} with initial conditions $u(x,0) = u_0(x)$.

This theorem is employed in proving the following theorem.

THEOREM 5.3. *Let $u^{n,\alpha}$ be solutions to the CFB equations with initial conditions $u^{n,\alpha}(x,0) = u_0^n(x)$. Let $u_0^n(x)$ converge to $u_0(x)$ in L^1 as $n \to \infty$. Let u^e be the entropy solution to the inviscid Burgers equation with initial conditions $u^e(x,0) = u_0(x)$. If Conjecture 5.1 holds true, then $u^{n,\alpha}$ converges to $u^e(x,t)$ in L^1_{loc} as $n \to \infty$ and $\alpha \to 0$ for any $u_0(x) \in L^\infty$.*

Proof. Let Ω be a compact subset of $\mathbb{R} \times [0,T]$. For $u^{n,\alpha}$ to converge to $u^e(x,t)$ in L^1_{loc},

$$(5.2) \qquad \lim_{n \to \infty \ \alpha \to 0} \iint_\Omega |u^{n,\alpha} - u^e| = 0.$$

Let $u^n(x,t)$ be the entropy solution to the inviscid Burgers equation with initial conditions $u^n(x,0) = u_0^n(x)$. We have assumed that

$$(5.3) \qquad \lim_{\alpha \to 0} \iint_\Omega |u^{n,\alpha} - u^n| = 0.$$

From Theorem 5.2 we know that

$$(5.4) \qquad \lim_{n \to \infty} \iint_\Omega |u^n - u^e| = 0.$$

Thus employing the triangle inequality we find

$$\lim_{n \to \infty \ \alpha \to 0} \iint_\Omega |u^{n,\alpha} - u^e| \leq \lim_{n \to \infty \ \alpha \to 0} \iint_\Omega |u^{n,\alpha} - u^n|$$

$$(5.5) \qquad\qquad + \lim_{n \to \infty \ \alpha \to 0} \iint_\Omega |u^n - u^e| = 0. \qquad \square$$

Using Theorem 5.3 it is easy to see that for the initial value problem

$$(5.6) \qquad\qquad u_t + (u * g^\alpha)u_x = 0,$$

$$(5.7) \qquad\qquad u(x,0) = u_0 * g^\alpha$$

the solutions will converge to the entropy solution of the inviscid Burgers equation with any initial condition $u_0(x)$ as $\alpha \to 0$. This scheme can handle discontinuous initial conditions, providing a greater usefulness.

6. Numerics. Section 5 proposes that (5.6) and (5.7) are a new system for the convectively filtered Burgers equation that is expected to converge to the entropy solution of the inviscid Burgers equation as $\alpha \to 0$ for all bounded initial conditions. This section runs some numerical simulation of the proposed system and shows evidence of convergence to the entropy solution.

6.1. The entropy solution. The specific initial condition being examined is the indicator function for the interval $(1,2)$ or

$$(6.1) \qquad\qquad u_0(x) = \begin{cases} 1 & \text{if } x \in (1,2), \\ 0 & \text{otherwise.} \end{cases}$$

For the entropy solution to the inviscid Burgers equation, the right-hand side of the initial pulse will form the standard right traveling shock and the left-hand side will form a rarefaction wave. At time $t = 2$, the rarefaction wave meets with the shock front, and then the shock front begins to decrease in amplitude and speed. For time $t < 2$ the entropy solution for the given initial conditions is

$$
(6.2) \qquad u(x,t) = \begin{cases} 0 & \text{if } x \leq 1, \\ \frac{x-1}{t} & \text{if } x \in (1, 1+t), \\ 1 & \text{if } x \in (1+t, 2+.5t), \\ 0 & \text{if } x \geq 2 + .5t. \end{cases}
$$

For time $t \geq 2$ the entropy solution is

$$
(6.3) \qquad u(x,t) = \begin{cases} 0 & \text{if } x \leq 1, \\ \frac{x-1}{t} & \text{if } x \in (1, (2t)^{\frac{1}{2}} + 1), \\ 0 & \text{if } x \geq (2t)^{\frac{1}{2}} + 1. \end{cases}
$$

It is to this solution that the CFB equations' solutions are compared.

6.2. Description of numerical methods. Holm and Staley performed successful simulations of the CFB equations with the Helmholtz filter using a pseudospectral method [31]. For this paper a very similar method is used. With the Helmholtz filter, (5.6) and (5.7) can be written as

$$
(6.4) \qquad \frac{\partial}{\partial t}\bar{u} + \frac{\partial}{\partial x}\frac{\bar{u}^2}{2} = -\frac{3}{2}\alpha^2 \left(I - \alpha^2 \left(\frac{\partial}{\partial x} \right)^2 \right)^{-1} \frac{\partial}{\partial x}(\bar{u}_x)^2,
$$

$$
(6.5) \qquad \bar{u}(x, 0) = (u_0 * g^\alpha) * g^\alpha.
$$

It is these equations that are numerically simulated.

Equation (6.4) is advanced through time with an explicit, Runge–Kutta–Fehlberg predictor/corrector (RK45). The initial timestep is chosen low enough to achieve stability and is then varied by the code using the formula

$$
(6.6) \qquad h_{i+1} = \gamma h_i \left(\frac{\varepsilon h_i}{||\bar{u}_i - \hat{u}_i||_2} \right)^{\frac{1}{4}}.
$$

Thus the new timestep is chosen from the previous timestep and the amount of error between the predicted velocity $\bar{u}$ and the corrected velocity $\hat{u}$. The relative error tolerance was chosen at $\varepsilon = 10^{-4}$ and the safety factor at $\gamma = 0.9$.

Spatial derivatives and the inversion of the Helmholtz operator were computed in the Fourier domain. The velocity was converted into the Fourier domain using a fast Fourier transform, multiplied by the appropriate term and then converted back into the physical domain. This pseudospectral method of calculating the derivative was chosen to reduce artificial viscosity.

In Holm and Staley's method, spatial derivatives were conducted using a fourth-order finite difference, and an artificial viscosity was applied to the high wave modes to prevent aliasing errors [31]. Because the simulations are addressing convergence to the entropy solution, as little artificial and numerical viscosity as possible is desired. For this reason derivatives were done in the Fourier domain and no artificial viscosity was introduced.
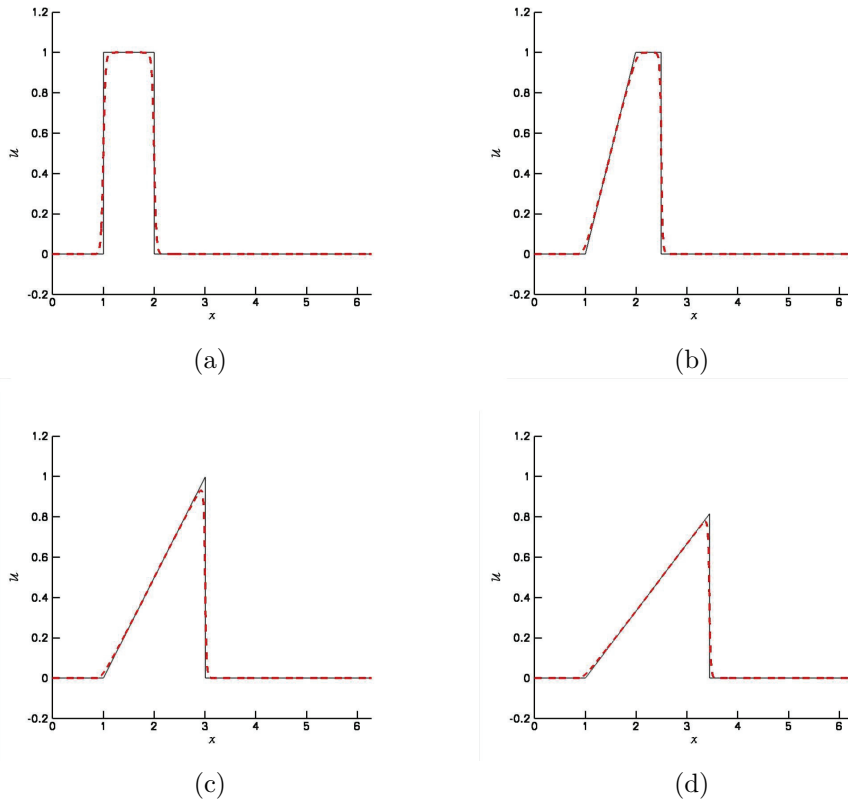
FIG. 6.1. *This figure compares the entropy solution with the solution to the CFB equations for* $\alpha = 0.02$. *It is easy to see that the CFB equations' solution is capturing both the rarefaction wave and the shock front behavior.*

The simulations were done at the resolution of $2^{12} = 4096$ grid points, thus being able to resolve 2048 Fourier modes. The two-thirds rule was utilized to prevent aliasing, thus reducing the effective resolution to 1364 Fourier modes. Despite this, there still appeared to be some long-term instability in the high wave modes. To control this instability, every 15 timesteps the higher two-thirds Fourier modes were zeroed which eliminated the instability and further reduced the effective resolution to 682 Fourier modes, which was acceptable for our purposes. It should be noted that simulations with the artificial viscosity have been also conducted and produce the same general results presented in the following sections. Additionally, simulations were conducted with a much higher resolution (2^{16}), where aliasing errors occurred, but remained small for the approximately 100,000 timesteps considered. These simulations also produced similar results.

6.3. Results. Nine different simulations were conducted with $\alpha = 0.02, 0.03, \ldots,$ 0.10. The CFB equations showed behavior mirroring that of the entropy solution. A traveling shock front and a rarefaction wave was seen. Figure 6.1 compares the CFB simulations for $\alpha = 0.02$ to the entropy solution at times $t = 0, 1, 2, 3$. In Figure 6.1(a) the difference in initial conditions can be seen with the entropy solution beginning with discontinuities and the CFB simulation having smoothed initial conditions.

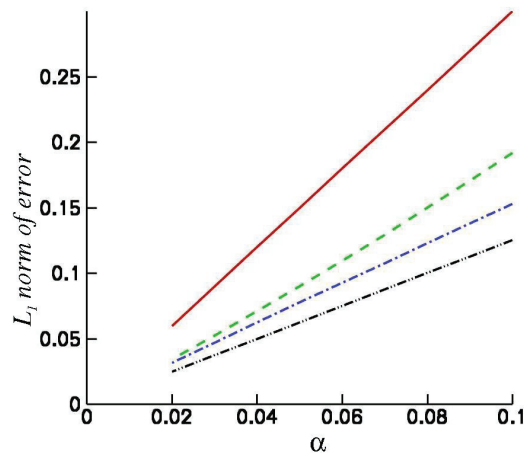To evaluate the convergence of the CFB equations' solutions to the entropy

FIG. 6.2. *The L_1 norm of the error between the CFB equations' solution and the entropy solution. The error is displayed for four different values of t. $t = 0$ ———, $t = 1$ – – – –, $t = 2$ –·–·–·–, and $t = 3$ –··–··–··. The error approaches zero roughly linearly as $\alpha \to 0$.*

solution, the L_1 norm of the error between the CFB equations' solution and the entropy solution was taken. Figure 6.2 plots α versus the error at times $t = 0, 1, 2, 3$. At each time the error appears to be approaching zero linearly. Thus numerical evidence suggests that (5.6) and (5.7) will converge to the entropy solution of the inviscid Burgers equation for initial conditions with discontinuities.

7. Conclusion. Conservation laws can often have multiple weak solutions of which there is one physically relevant solution, known as the entropy solution. It is important that any regularization of these conservation laws reflect the physical phenomenon they are meant to address. Thus it is important that the solutions to such regularizations converge to the entropy solution. The convectively filtered Burgers equation has been shown to regularize the inviscid Burgers equation. This paper now shows that for a certain class of initial conditions this regularization will converge to the entropy solution. It has also provided a method for extending this convergence to a large class of initial conditions including discontinuities. These results are a crucial step in extending the convectively filtered method into popular use and perhaps into the Euler equations.

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