Statistical equilibrium theory for axisymmetric flows: Kelvin's variational principle and an explanation for the vortex ring pinch-off process

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Thermodynamics of vorticity density fields (ω/r) in axisymmetric flows are considered, and the statistical equilibrium theories of Miller, Weichman, and Cross [Phys. Rev. A **45**, 2328 (1992)], Robert and Sommeria [J. Fluid Mech. **229**, 291 (1991)], and Turkington [Comm. Pure Appl. Math. **52**, 781 (1999)] for the two-dimensional flows in Cartesian coordinates are extended to axisymmetric flows. It is shown that the statistical equilibrium of an axisymmetric flow is the state that maximizes an entropy functional with some constraints on the invariants of motion. A consequence of this argument is that only the linear functionals of vorticity density, e.g., energy and total circulation, are conserved during the evolution of an axisymmetric inviscid flow to the statistical equilibrium. Furthermore, it is shown that the final equilibrium state satisfies Kelvin's variational principle; the mean field profiles maximize the energy compatible with the resulting dressed vorticity density. Finally, the vortex ring pinch-off process is explained through statistical equilibrium theories. It appears that only a few invariants of motion (the kinetic energy, total circulation, and impulse) are important in the pinch-off process, and the higher enstrophy densities do not play a significant role in this process. © *2001 American Institute of Physics.*
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I. INTRODUCTION

It is well known that the evolution of two-dimensional (2D) plane turbulence is dominated by the formation of coherent vortices, see, e.g., Refs. 1 and 2. The formation of coherent structures in freely decaying 2D flows eventually results in a quasisteady state; the velocity field becomes increasingly dominated by the larger spatial scales as time progresses, and the like-signed vortex regions merge into increasingly larger vortices.[2] Many numerical and experimental studies suggest that the formation of large scale structures is mainly an inviscid process, and that the viscosity and dissipation only affect the fine scale motion. These studies lead to the idea of using statistical mechanics to understand these long-lasting structures in 2D incompressible flows.

The first insight into the equilibrium properties of 2D flows as a Hamiltonian system, was provided by Onsager's statistical theory of point vortices,[3] which were later extended by Joyce and Montgomery.[4] For high enough Reynolds numbers, the enstrophy can decay significantly while the energy is decaying by a negligible amount. Onsager predicted the formation of large scale vortices (regions of vortices of the same sign) in 2D Euler equations. For vortices in a confined region, Onsager noticed that contrary to statistical thermodynamics, the temperature could be either negative or positive. Therefore, for the negative temperatures the statistical probability of observing high energy states is higher than for low energy states. The high energy states normally correspond to clustering vortices with the same sign, dubbed as coherent structures. There are some criticisms of point vortex models: The infinite conservation laws of moments of vorticity are not respected, the quantitative prediction depends on the representation of a continuous vorticity field in terms of point vortices (which is not unique), and the maximum vorticity is not bounded by its initial maximum, as it should be for 2D flows.

In a recent work, Lim[5,6] derived a long range spherical model for Kirchoff–Helmholtz vortex gas,[7] and subsequently the 2D Euler equations. He showed that a certain spin-lattice Hamiltonian removes the nonuniqueness ambiguity from the mean field theory of Onsager[3] and Joyce and Montgomery.[4] He exactly solved the statistical equilibrium theory of energy–enstrophy for the barotropic vorticity equation in the sense that an explicitly non-Gaussian configuration integral was calculated in a closed form.

To remove some of the limitations of theories based on point vortices, Miller, Weichman, and Cross[8,9] (MWC) and independently Robert and Sommeria (RS)[10] developed a mean field theory. Their resulting equilibrium state is calculated by maximizing a mixing entropy constrained by the invariants of motion, which are the energy, impulse, and the global probability distribution of the vorticity fluctuations. A main criterion for the validity of their approach in predicting the long term behavior of high Reynolds number flows is that the relaxation time for the system be shorter than the viscous time scale. When this condition is satisfied the system relaxes to an almost equilibrium state, before the viscous effects alter the integrals of higher moments of vorticity. This raises the speculation that only a finite number of constraints might be sufficient for a useful equilibrium theory of

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2D flows, e.g., see Chorin[11] and Chavanis and Sommeria.[12] It has been noted that the statistical equilibrium theory of MWC–RS works remarkably well in the case of vortex merging and for large scale initial vorticity fields where the relaxation is relatively violent and take place in a few eddy turnover times. On the other hand, when the initial condition has significant small scale contributions the relaxation to equilibrium takes much longer time and viscosity may alter the invariants of motion.[13–15] Hence, strong discrepancies can be observed when a prediction from the initial vorticity field is made. However, it was pointed out that if the constants of motion are calculated from the vorticity field at later times the agreement between the entropy maximization and the quasistationary state of a weakly viscous dynamics improves.[13] To address Chorin's speculation,[11] Turkington[16] and Boucher *et al.*[17] recently developed a statistical theory with a few constraints, where they replaced the equality constraints on the general vorticity integrals in the MWC–RS theory with inequality constraints. Some indications on the validity of their idea in the vortex ring formation was considered in Sec. IV of this study.

In this paper we are concerned with axisymmetric flows, one of the simplest three-dimensional flows. Although axisymmetricity is a limitation, a wide range of challenging problems reside in this category, including jet flows, vortex rings, drops, and pipe flows. The severe difference between the 2D and 3D turbulence is usually contributed to the vanishing of the vortex stretching term in 2D flows. While the general vortex stretching term in the vorticity equation is missing in the axisymmetric flows as well, the existence of a geometrical stretching term makes it more interesting than 2D Cartesian flows.

Our main motivation for the study of long time behavior of axisymmetric flows comes from our interest in understanding the universal formation number of vortex ring pinch-off processes observed in experiments by Gharib *et al.*,[18] theoretical modeling by Mohseni and Gharib,[19] and the numerical simulations of Navier–Stokes equations by Mohseni *et al.*[20] In the laboratory, vortex rings can be generated by the motion of a piston pushing a column of fluid through an orifice or nozzle. The boundary layer at the edge of the orifice or nozzle will separate and roll up into a vortex ring. We think that since the formation of vortex rings involves strong mixing of the generated shear layer with the ambient fluid, the ergodicity requirement of statistical equilibrium theory has a chance to be satisfied. The experiments of Gharib *et al.*[18] have shown that for large piston stroke versus diameter ratios (L/D), the generated flow field consists of a leading vortex ring followed by a trailing jet. The vorticity field of the formed leading vortex ring is disconnected from that of the trailing jet at a critical value of L/D (dubbed the ''formation number''), at which time the vortex ring attains a maximum circulation. The formation number was in the range of 3.6–4.5 for a variety of exit diameters, exit plane geometries, and nonimpulsive piston velocities. An explanation for this phenomenon was given based on Kelvin's variational principle. It was both experimentally[18] and analytically[19] observed that the limiting stroke L/D occurs when the generating apparatus is no longer able to deliver energy, circulation, and impulse at a rate comparable with the requirement that a steadily translating vortex ring has maximum energy with respect to kinematically allowable perturbations. As demonstrated in Sec. IV, Kelvin's variational principle (energy extremization) has a close connection with the entropy maximization in statistical equilibrium theory. Numerical evidence for a relaxation process to an equilibrium state has already been provided by Mohseni *et al.*[20] in a direct numerical simulation of the pinch-off process in vortex ring formation.

An interesting observation in this paper is the consistency between Kelvin's variational principle, dressed vorticity density corollary, Turkington's approach, and the experimental and numerical observations that the first few invariants of motion, namely the energy, impulse, and circulation, play the most significant role in the vortex ring pinch-off process. To this end, our results support Chorin's[11] and Turkington's[16] idea of a mean field theory with a few constraints. In this paper we take the first step in explaining the vortex ring pinch-off process through a statistical equilibrium theory. The theoretical foundation has been laid out in this paper and the details of the numerical experimentation on the mean field equations will be postponed to a future publication.

Our objectives in this study are several. Following Szeri and Holmes[21] (also see Mohseni[22]), in Sec. II we derive an explicit expression for a canonical Poisson bracket of axisymmetric flows which is similar to the Poisson bracket of the 2D plane case. This Poisson bracket satisfies the Jacobi identity (among other properties), and therefore makes the space of functions of vorticity density fields on Ω (the volume occupied by the fluid) into a Lie algebra. In Sec. III our goal is to ask whether we can predict and explain the long-time evolution of flows, such as those mentioned above, without explicitly using dynamics. Costly dynamical simulations and significant errors at long times make such a theory attractive for investigating problems where the transient dynamics are not of primary interest. In Sec. IV we show that Kelvin's energy variational result can be deduced from the statistical equilibrium equations. An explanation of the vortex ring pinch-off process as a relaxation of an axisymmetric vortical system to its final equilibrium state, predicted by the statistical equilibrium theory, is also considered in this section. Finally, the concluding remarks are presented in Sec. V.

II. GOVERNING EQUATIONS AND POISSON BRACKET

In this section we study the Hamiltonian structure of axisymmetric flows. It is not obvious how to develop a statistical mechanics theory without a Hamiltonian. Once the Hamiltonian is given, very few choices in the development of the theory remain.

Consider an axisymmetric, inviscid, homogeneous, and incompressible flow in a 3D axisymmetric region Ω. The velocity $\mathbf{u}(u_r,0,u_x)$ of this flow is governed by the vorticity evolution equation

$$\frac{\partial\omega}{\partial t}+u_x\frac{\partial\omega}{\partial x}+u_r\frac{\partial\omega}{\partial r}=\frac{u_r\omega}{r}, \tag{1}$$

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$$\omega=(\Delta\times\mathbf{u})_\phi=\frac{\partial u_r}{\partial x}-\frac{\partial u_x}{\partial r}. \tag{2}$$

The scalar ω is the azimuthal component of vorticity. The $u_r\omega/r$ term on the right-hand side of the vorticity equation (1) is the geometrical vortex stretching. This term is absent in the 2D vorticity equation in Cartesian coordinates.

The governing system consists of a transport equation (1) coupled with the elliptic system (2) and the continuity equation $\nabla\cdot\mathbf{u}=0$. We would like to use a formulation in terms of the ''vorticity density'' ξ defined as $\xi=\omega/r$. At any instant, the velocity field can be described by the Stokes stream function ψ defined by

$$u_x=\frac{1}{r}\frac{\partial\psi}{\partial r};\quad u_r=-\frac{1}{r}\frac{\partial\psi}{\partial x}.$$

Using the transformation $y=r^2/2$, one may recast the final set of equations as

$$\frac{D\xi}{Dt}:=\frac{\partial\xi}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial\xi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\xi}{\partial y}=0, \tag{3}$$

and

$$\mathcal{L}(\psi):=\frac{\partial^2\psi}{\partial y^2}+\frac{1}{2y}\frac{\partial^2\psi}{\partial x^2}=-\xi. \tag{4}$$

A suitable phase space for this system is the space of ξ's defined on the physical space occupied by the fluid. The natural choice for the Hamiltonian is the kinetic energy (with unity density) given by

$$H=\frac{1}{2}\int_{R^3}\mathbf{u}^2\,dV, \tag{5}$$

where $dV=r\,dr\,dx\,d\theta$ represents the volume element. In the case of axisymmetric flows without swirl, one can write

$$\begin{aligned}H&=\pi\int_{\Omega_r}(u_x^2+u_r^2)r\,dx\,dr\\&=\pi\int_{\Omega_r}\left(u_x\frac{\partial\psi}{\partial r}-u_r\frac{\partial\psi}{\partial x}\right)dx\,dr\\&=\pi\int_{\Omega_r}\left(\omega\psi+\frac{\partial(u_x\psi)}{\partial r}-\frac{\partial(u_r\psi)}{\partial x}\right)dx\,dr\\&=\pi\left\{\int_{\Omega_r}\omega\psi dx\,dr-\oint\psi(u_xdx+u_r\,dr)\right\}\\&=\pi\left\{\int_{\Omega_y}\xi\psi dx\,dy-\oint\psi(u_x\,dx+u_r\,dr)\right\},\end{aligned} \tag{6}$$

where Ω_r is the half space $r\geqslant 0$, and Ω_y is the half space $y\geqslant 0$. By appropriate assumptions on the behavior of the flow field near the boundaries we can ignore the second term on the right-hand side of Eq. (6). Therefore, we can write the kinetic energy (also called the excess kinetic energy[23]) as

$$H=\pi\int_{\Omega_y}\xi\psi d\mu=\pi\int\int\mathcal{M}(\mathbf{x}|\mathbf{x}')\xi(\mathbf{x})\xi(\mathbf{x}')d\mu\,d\mu', \tag{7}$$

where $x=xi_x+yi_y$, $d\mu=dx\,dy$ is the area element in Ω_y, and the kernel $\mathcal{M}(\mathbf{x}|\mathbf{x}')$ is defined by[24]

$$\mathcal{M}(\mathbf{x}|\mathbf{x}')=\frac{\sqrt{yy'}}{2\pi}\int_0^{2\pi}\frac{\cos\theta\,d\theta}{\sqrt{(x-x')^2+2y+2y'-4\sqrt{yy'}\cos\theta}}. \tag{8}$$

Clearly the kinetic energy is conserved by the flow. Now, we define the Lie–Poisson bracket

$$\{F,G\}=\int\xi(\mathbf{x})\left\{\frac{\delta F}{\delta\xi},\frac{\delta G}{\delta\xi}\right\}_{xy}d\mu, \tag{9}$$

where

$$\{f,g\}=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

is the canonical (x,y) Poisson bracket. One may use (9) to show that

$$\xi_t=\{\xi,H\}, \tag{10}$$

which is another form of the vorticity density evolution equation. It is not known whether there exists a pair of canonical coordinates (functionals of ξ) which diagonalize the bracket (9). However, what is required in statistical equilibrium theory is an invariant measure on the phase space, which will be provided by Liouville's theorem. Therefore, neither the existence of the canonical coordinate nor the existence of the Lie–Poisson bracket (9) is required in the statistical equilibrium theory. The dynamics of the Euler equations preserves phase space volumes, i.e., the flow in the phase space is incompressible. The ergodic hypothesis ascribes weights in proportion to phase space volume. Therefore, the phase space flow preserves relative probabilities in the chosen variables. It also follows that the new radial variable y is necessary to compensate for the geometrical increase in the volume element of the $x-r$ space; $dx\,dy=r\,dx\,dr$. Finally, we would like to note that similar results have been obtained by Szeri and Holmes[21] in considering the nonlinear stability of axisymmetric swirling flows.

Equation (3) implies that there are an infinite number of conserved vorticity density integrals, $\int_{\eta(t)}f(\xi)d\mu$, for any path $\eta(t)$ moving with the fluid where f is an arbitrary function. Therefore, one can define an infinite number of conserved quantities for the Euler equation (3) which can be characterized by

$$I_n=\int_{\Omega_y}\xi^n d\mu. \tag{11}$$

Following the form of the Lie–Poisson bracket (9) it follows that these conserved quantities are Casimirs. Finally, one can verify that the linear momentum

$$P=\frac{1}{2}\int(\mathbf{x}\times\boldsymbol{\omega})_x\hat{\theta}\,dV=2\pi\int_{\Omega_y}\xi y\,d\mu \tag{12}$$

is also an invariant of motion.

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The Poisson bracket (9) that we have derived for axisymmetric ideal flows is similar to the canonical Poisson bracket of the 2D plane flows.[25] It satisfies the Jacobi identity (inherited from that for $\{,\}_{xy}$, as is generally the case for Lie–Poisson brackets), among other properties. Therefore, Ω_y is a Lie algebra. One might expect to obtain the same results by a reduction process[25,26] from the 3D Euler equations with axial symmetry.

The Hamiltonian structure of axisymmetric vortex filaments has been known for more than a century, see, e.g., Shariff *et al.*[27] and references therein. The Hamiltonian formalism of vortex filaments can be considered as a special case of the field Hamiltonian formalism of this section. We refer to the Appendix (see also Mohseni[28]) for a derivation of the Hamiltonian formalism of axisymmetric vortex filaments from the field Hamiltonian. Now that the Hamiltonian formalism for axisymmetric flows is established, we can proceed with our development of statistical mechanics of vorticity density fields.

III. STATISTICAL EQUILIBRIUM THEORY

In this section we will extend the statistical equilibrium theories of MWC–RS and Turkington for 2D plane flows to axisymmetric flows. In doing so we assume ergodicity, i.e., that the dynamics of the flow samples all the phase space consistent with the conservation laws. The treatment we describe is to cast the problem as a variational problem with constraints. The maximization of an entropy functional will then result in the equations for statistical equilibrium of the flow. The derivation technique is basically the same for both MWC–RS and Turkington's theories. Therefore, we first extend MWC–RS theory to axisymmetric flows. By relaxing the equality constraints on the generalized vorticity density to inequalities, we can then obtain the maximization problem for Turkington's method.

The governing equations are given by (3) and (4), and the conserved quantities of the axisymmetric Euler equations are H, P, and the infinite series I_n. The conservation of P results from the absence of global pressure forces in the x direction. Note that H and I_n are conserved in the general case for any domain Ω. By contrast the conservation of the physical momentum P is specific to a given geometry.

In the axisymmetric Euler equation (3), the vorticity density ξ is a material property. This implies that the total area fraction occupied by each vorticity density level σ is conserved. We call this quantity the global probability distribution of vorticity density $\gamma(\sigma)$. The distribution function $n_0(\mathbf{x},\sigma)$ is defined as the local (but coarse-grained) probability of finding the vorticity level σ in a $\mathbf{x}$ neighborhood. Therefore, $n_0(\mathbf{x},\sigma)$ may be considered as the local density of the vorticity density with strength σ. This distribution function satisfies the normalization condition (incompressibility)

$$\int n_0(\mathbf{x},\sigma)\,d\sigma = 1. \tag{13}$$

Subsequently, (11) may be recast as the conservation of $\gamma(\sigma)=\int n_0(\mathbf{x},\sigma)\,d\mu=\int \delta(\sigma-\xi(\mathbf{x}))\,d\mu$. Physically, $\gamma(\sigma)$ measures the fractional area covered by the vorticity level σ. Now the coarse-grained (macroscopic) equilibrium vorticity density $\overline{\xi}(\mathbf{x})$ is defined as $\overline{\xi}(\mathbf{x})=\int n_0(\mathbf{x},\sigma)\sigma\,d\sigma$. The coarse-grained equilibrium stream function $\overline{\psi}$ is related to $\overline{\xi}$ through Eq. (4). Note that the macrostates have fluctuations in vorticity density, but as $\overline{\psi}$ is obtained by an integration of vorticity density, there are no fluctuations in the stream function. For a system with very small viscosity, we expect that these fluctuations will be smoothed out, so that the resulting steady flow becomes the actual final state of the system. Finally, the entropy is defined by

$$S=-\int n_0(\mathbf{x},\sigma)\ln n_0(\mathbf{x},\sigma)\,d\sigma\,d\mu. \tag{14}$$

Now the standard methods of statistical mechanics can be used. We started from a Hamiltonian system which gives the dynamics of a great number of particles and which is the microscopic level of description of the system. Then, at a macroscopic level, we consider some relevant means, which we call macroscopic observable. To these two levels of description we associate an entropy functional using Boltzmann's formula $S=k\log W$, where W is the volume occupied in the phase space (endowed with the invariant Liouville measure) by the set of all the microstates giving the same macrostate. Maximizing the entropy functional then gives the equilibrium states. The entropy functional in Eq. (14) represents the logarithm of the number of possible vorticity configurations associated with a final macrostate.

The equilibrium state is obtained by maximizing the mixing entropy, (14) subjected to the constraints of motion (11)–(12) and the normalization condition (13). We write these constraints in terms of the locally averaged vorticity density $\overline{\xi}$ and the associated stream function $\overline{\psi}$. The resulting constrained variational problem can be treated by introducing the Lagrange multipliers such that the first variations satisfy

$$\delta S-\beta\delta H-\beta\int \alpha(\sigma)\delta\gamma(\sigma)\,d\sigma - \int \zeta(\mathbf{x})\,\delta\!\left(\int n_0\,d\sigma\right)d\mu-\beta U\delta P=0, \tag{15}$$

where β is the inverse of the temperature, $\alpha(\sigma)$ is the chemical potential of species σ, and U is the translation velocity. In general, energy or temperature may be considered as an indicator of how closely the vorticity is packed (we will come back to this point later). The final state can be considered either as a critical point of the entropy for any admissible perturbations (that satisfies the invariants of motion) or equivalently, as a critical point of the free energy F[29]

$$F=S-\beta H-\beta\int \alpha(\sigma)\gamma(\sigma)\,d\sigma - \int\int \zeta(\mathbf{x})n_0\,d\sigma\,d\mu-\beta UP. \tag{16}$$

The resulting distribution function is a Gibbs state of the form

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$$n_0(\mathbf{x},\sigma)=e^{-\beta[2\pi\sigma(\overline{\psi}+yU)+\alpha(\sigma)]-1-\zeta}. \tag{17}$$

Using the normalization constraint (13) we can remove the Lagrange multiplier ζ to obtain the resulting distribution function

$$n_0(\mathbf{x},\sigma)=\frac{e^{-\beta[2\pi\sigma(\overline{\psi}+yU)+\alpha(\sigma)]}}{\int e^{-\beta[2\pi\sigma(\overline{\psi}+yU)+\alpha(\sigma)]}d\sigma}. \tag{18}$$

A differential equation for $\overline{\psi}$ may be obtained by multiplying both sides of (18) by σ and integrating

$$\mathcal{L}(\overline{\psi}):=\frac{\partial^2\overline{\psi}}{\partial y^2}+\frac{1}{2y}\frac{\partial^2\overline{\psi}}{\partial x^2}=\frac{1}{2\pi\beta}\frac{d}{d\overline{\psi}}\ln\mathcal{Z}, \tag{19}$$

where the partition function $\mathcal{Z}$ is given by

$$\mathcal{Z}(\overline{\psi})=\int e^{-\beta[2\pi\sigma(\overline{\psi}+yU)+\alpha(\sigma)]}d\sigma. \tag{20}$$

The partition function relates the properties of the microscopic system, and it provides all the statistical parameters of a macroscopic system. The Lagrange multipliers β, U, and functions $\alpha(\sigma)$ are determined by the initial conditions, namely

$$\gamma(\sigma)=\int\frac{e^{-\beta[2\pi\sigma(\overline{\psi}+yU)+\alpha(\sigma)]}}{\mathcal{Z}(\overline{\psi}(\mathbf{x}))}d\mu,$$

$$E=-\pi\int\overline{\psi}\mathcal{L}(\overline{\psi})d\mu, \tag{21}$$

$$P=-2\pi\int y\mathcal{L}(\overline{\psi})d\mu,$$

the global conservation of vorticity density $\gamma(\sigma)$, energy E, and the linear momentum P, respectively. The equilibrium states are not steady in general but translate uniformly with velocity U. In the case of nonvanishing circulation, we can change the frame of reference to the one moving with the center of vorticity density, where $P=0$. The stream function in this case is obviously $\psi+yU$. Knowing the invariants of motion and the Lagrange multipliers, Z will be only a function of $\overline{\psi}$, and therefore, Eq. (19) will be of the form $\mathcal{L}(\overline{\psi})=f(\overline{\psi})$.

Chorin[11] recently indicated that a few constraints might be sufficient for a reasonable theory of statistical equilibrium. Subsequently, Turkington *et al.*[16,17] presented a statistical equilibrium theory for the 2D Euler equations based on a few constraints, where they criticized the implicit assumption in the MWC–RS theory that the microstate vorticity on the lattice satisfies the same constraints as the vorticity solutions to the Euler equations in the physical domain. Consequently, in Turkington's model the family of enstrophy constraints relaxes to inequalities. The diverging point between the MWC–RS technique and Turkington's method is the calculation of the mean vorticity distribution from the generalized enstrophy density and the prior choice of the probability measure.

Turkington's method can be easily extended to the axisymmetric flows of this study by some modifications of the theory developed in this section. It can be easily shown that, unlike H, the generalized enstrophy density I_n depends on the smallest vorticity fluctuations. In our extension of the MWC–RS theory to axisymmetric flows the generalized enstrophy densities were treated the same as the total kinetic energy H. However, in extending the Turkington *et al.*[16,17] model to axisymmetric flows the equality constraints on the generalized enstrophy density in the axisymmetric maximum entropy principle is relaxed to a family of convex inequality constraints. This approach is compatible with the experimental observations and numerical simulations of the vortex ring pinch-off process discussed in Sec. IV.

IV. KELVIN'S VARIATIONAL RESULT AND VORTEX RING PINCH-OFF PROCESS

Kelvin's variational principle was recently applied in explaining the vortex ring pinch-off process.[18] In this section we investigate the relation between Kelvin's variational principle and the statistical equilibrium theories of Sec. III. This will offer another explanation of the vortex ring pinch-off process based on the statistical equilibrium theory.

A general variational principle due to Kelvin[30] characterizes steady flows in a 2D ideal fluid as the stationary values of the kinetic energy for given circulation and hydrodynamic impulses, with respect to kinematically allowable perturbations. Kelvin states the principle without proof as being obvious to him (see Secs. 4 and 18 of Kelvin[30]). The conceptual basis of this theory has been furnished by Benjamin,[31] whose work is especially noteworthy because it connects the abstract variational principles with concrete model problems in ideal fluid dynamics. Conceptually, this variational result leads to a formulation of the general mathematical problem entirely in terms of the natural physical invariants associated with the equations governing vortex dynamics: energy, impulse, and circulation. A one-parameter family of such solutions was presented by Norbury.[32] Wan[33] studied the maximization property of a limiting case in Norbury families, namely Hill's spherical vortex. Both Kelvin's approach and the statistical equilibrium theory are of variational types concerning the equilibrium states of Euler equations. While Kelvin prefers extremization of an energy functional, in statistical equilibrium theory it is the maximization of a mixing entropy that determines the final equilibrium state. Consequently, one might expect that a close relation exists between the final equilibrium states predicted by these two approaches. In this section we show that the equilibrium solution predicted from the statistical equilibrium theory (entropy maximization) satisfies an energy extremization similar to Kelvin's approach with a few explicit constraints.

An input to the statistical theories of the previous section is the initial vorticity density distribution $\xi(\mathbf{x})$, or equivalently the invariants of motion in the form of Casimirs. However, in most practical applications (e.g., the Red Spot of Jupiter and vortex ring pinch-off process[19,20]) our information on the initial condition is very limited. What is usually measurable is the finite resolution vorticity distribution $\overline{\xi}(\mathbf{x})$.

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Therefore, the only measurable distribution function is the dressed distribution function (as defined for the 2D case in Ref. 9)

$$\gamma_d(\sigma) = \int \delta(\sigma - \overline{\xi}) d\mu. \tag{22}$$

Here, $\gamma(\sigma)$ is the initial distribution function and $\gamma_d(\sigma)$ is the distribution function observed on any finite length scale. Note that $\overline{\xi}(\mathbf{x})$ is a smooth function except for $\beta \to \pm\infty$ and that in general the vorticity distribution function $\gamma_d(\sigma)$ derived from the mean field profile $\overline{\xi}$ is not the same as $\gamma(\sigma)$. In the process of statistical equilibrium, when going from the microscopic description to the macroscopic description, it is natural that a major part of the information about the details of the small scales is lost. Therefore, the vorticity density conservation laws (11), except for the total circulation I_1, are all violated on the macroscopic scale. No other moment of the vorticity density is necessarily the same for both γ and γ_d. In this process only the energy, circulation, and impulse are conserved both on the fine scales and on the coarse scales. In general, when using the final vorticity distribution from the statistical equilibrium theory, only linear functionals of vorticity are conserved. Although it is impossible to experimentally infer $\gamma(\sigma)$ from the equilibrium state alone, one can make partial predictions by knowing $\gamma_d(\sigma)$. This is, in fact, consistent with the observations in 2D Euler or high Reynolds number Navier–Stokes equations, where it is well known that there is an inverse cascade of energy to large scales and a forward cascade of enstrophy (second integral of vorticity I_2) to smaller scales. Therefore, by measurement of finite resolution one might expect to recover almost all of the initial kinetic energy of the system, while the conservation of enstrophy will be violated.

Following Miller *et al.*,[9] a *dressed vorticity density corollary* is in order: $\overline{\xi}(\mathbf{x})$, the averaged vorticity density field, is the maximum energy solution (corresponding to $T \to 0^-$ or $\beta \to -\infty$) of the statistical equilibrium equations with constraint function $\gamma_d(\sigma)$ (note that the negative temperatures correspond to the clustering of vortices with the same sign). For the maximum energy solution, $\gamma_d(\sigma) = \gamma(\sigma)$. The proof is analogous to the proof for a similar corollary in 2D turbulence in the plane[9] and it is not repeated here. Although γ_d is in general different from γ, a consequence of the above argument is that at a given energy, γ_d results in the same equilibrium solution as γ. Furthermore, the given energy turns out to be precisely the maximum energy compatible with γ_d.

Note that aside from the total kinetic energy and hydrodynamic impulse the total circulation I_1 is also preserved during the equilibrium process, i.e.,

$$\int \sigma\gamma(\sigma) d\sigma = \int \sigma\gamma_d(\sigma) d\sigma. \tag{23}$$

However, in general the conservation of any other generalized enstrophy density integrals I_n would be violated. Now the connection between Kelvin's variational principle and the dressed vorticity density corollary is clear: for a system with fixed circulation and impulse the statistical equilibrium state is the one that maximizes the energy consistent with the $\gamma_d(\sigma)$, i.e., its isovortical perturbations. Note that the higher enstrophy densities are not explicitly specified in Kelvin's variational principle (in contrast to Arnold's approach[34]). Since the energy, circulation, and impulse of the system are conserved during the equilibrium process, the final solution of the statistical equilibrium theory (or equivalently dressed vorticity density corollary) satisfies the requirements of Kelvin's variational principle. It is interesting to note that the conservation of energy, circulation, and impulse have a consequential effect on the dressed vorticity density corollary, Kelvin's variational principle, and Turkington's model, while the higher generalized enstrophy densities seem to be insignificant in the final state of the system. The connection between the equilibrium states predicted by the statistical equilibrium theories and the steady state solutions of the Euler equations is clear in this argument.

An implication of the dressed vorticity corollary is that for a fluid in statistical equilibrium, coarse-grained quantities suffice to determine the equilibrium. Chavanis and Sommeria[12] showed that in the limit of strong mixing the higher enstrophy densities do not have a significant effect on the final equilibrium state predicted in the theory of MWC–RS. This observation is also consistent with Turkington's theory.[16] Therefore, it is expected that our equilibria might persist in the presence of a viscosity acting to smear the small scales. An equivalent way of stating this result is that the long-time dynamics of an inviscid fluid will evolve to a configuration which is a global extremum of the energy, subject to satisfying the long-time (dressed) vorticity distribution.

Our motivation for studying axisymmetric flows comes from our interest in the formation of coherent vortical structures in jets and at the exit of nozzles. An interesting problem, in this context, is the pinch-off process in vortex ring formation at the exit of a nozzle in cylinder–piston mechanism; see, e.g., Gharib *et al.*[18] for experimental observations, Mohseni and Gharib[19] for modeling, and Mohseni *et al.*[20] for computational results. For large piston stroke versus diameter ratios (L/D), the generated flow field consists of a leading vortex ring followed by a trailing jet. At a critical value of L/D (dubbed the ''formation number''), at which time the vortex ring attains a maximum circulation, the vorticity field of the formed leading vortex ring is disconnected from that of the trailing jet. An explanation for this phenomenon was given based on Kelvin's variational principle. It was both experimentally[18] and analytically[19] observed that the limiting stroke L/D occurs when the generating apparatus is no longer able to deliver energy, circulation and impulse at a rate comparable with the requirement that a steadily translating vortex ring has maximum energy with respect to kinematically allowable perturbations. The formation number was observed[18] to be in the range 3.6–4.5 for a variety of exit diameters, exit plane geometries, and nonimpulsive piston velocities.

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Inspired by these observations we offer a relaxational (statistical approach to the pinch-off process.[19,20] This is an alternative explanation of the vortex ring pinch-off process, based on a mixing entropy maximization, besides the energy extremization approach in Kelvin's variational principle. From this point of view, any vortex ring generator can be viewed as a tool for initializing an axisymmetric flow with a particular rate of the generation of invariants of motion. Each vortex ring generator has a specific rate for feeding the flow with the kinetic energy, impulse, circulation, etc. In this picture, at small strokes (small L/D) one will find that all of the initial vorticity density will coalesce into a steadily translating vortex ring. As the stroke increases the size, strength, and the translational velocity of the resulting vortex ring increase. This process persists until a critical formation number is reached, when the vortex generator is not able to provide invariants of motion compatible with a single translating vortex ring. Equivalently, beyond the critical formation number a single vortex ring at equilibrium (steadily translating) that maximizes the mixing entropy for given energy, impulse, and circulation is not possible. In this case the leading vortex ring will pinch-off from the trailing jet and will relax to a translating vortex ring with the translational velocity U dictated in the maximum entropy principle. For very large strokes (greater than twice the critical formation number) successive vortex rings will pinch-off from the the trailing jet. This scenario was verified in the numerical simulations of the vortex ring pinch-off process in Mohseni *et al.*[20] The general observation in these simulations was that the main invariants of motion in the pinch-off process are the kinetic energy, circulation, and impulse. The higher enstrophy densities *did not* play a significant role as long as the Reynolds number was relatively high. These observations confirm Chorin's[11] and Turkington's argument[16] that a statistical equilibrium theory with a few constraints might be enough for an accurate prediction of the equilibrium states.

V. CONCLUSIONS

The equations derived in this investigation give relaxed end-states of the axisymmetric Euler equations and are believed to be closely related to the forced and slightly damped turbulent dynamics of Navier–Stokes equations.[7] Costly dynamical simulations based on the Euler or Navier–Stokes equations and significant errors at long times make such a theory attractive for investigating problems where the transient dynamics are not of primary interest. The resulting solutions of these statistical equilibrium theories are in fact the equilibrium solutions to the axisymmetric Euler equations, constrained by the invariants of motion. We observed that while the infinite number of Casimirs (enstrophy densities) were important in the development of the theory, integrals of the nonlinear powers of vorticity density measured on any physical scale will not be the same as in the initial state and can be relaxed as was suggested by Turkington *et al.*,[16,17] Chorin,[11] and Chavanis and Sommeria.[12] The only apparent conserved quantities are the energy, impulse, and total circulation. In fact, these conserved quantities are explicitly specified in Kelvin's variational result. Through a dressed vorticity density corollary the connection between the statistical equilibrium state and Kelvin's variational principle was investigated. The maximum entropy state predicted by the statistical equilibrium theories satisfies Kelvin's variational principle (energy extremization).

Statistical relaxation of axisymmetric flows offers an explanation for the vortex ring pinch-off process:[18–20] the system relaxes to an equilibrium state from its initial configuration dictated by the stroke ratio L/D in the cylinder–piston mechanism of vortex ring generation or more generally by TU/h for a general vortex ring generator. Here T is the period of vortex shedding, U is the translational velocity, and h is the toroidal radius of the resulting vortex ring. It was suggested that the final state of the problem is governed by the first few invariants of motion, namely the energy, impulse, and circulation. Note that any vorticity generation mechanism has its own specific rate for the generation of these invariants of motion. For a cylinder piston mechanism these rates are given in Mohseni and Gharib.[19] The physical explanation is that for short strokes the system relaxes to a small steadily translating vortex ring. Increasing the stroke results in a larger vortex ring. For high enough strokes (above the formation number) the traditional cylinder piston mechanism is not able to provide energy compatible with an equilibrium state at the same circulation and impulse that maximizes the mixing entropy in the statistical equilibrium theory. This is an alternative explanation, besides the energy extremization in Kelvin's variational principle, for the vortex ring pinch-off process.

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APPENDIX: HAMILTONIAN FORMALISM OF AXISYMMETRIC VORTEX FILAMENTS

We assume that the vorticity density distribution can be approximated by a combination of discrete axisymmetric vortex filaments as

$$\xi(\mathbf{x}) = \sum_{i=1}^{N} c_i \delta(\mathbf{x} - \mathbf{x}_i).$$

Substituting this equation into the Hamiltonian (7) we obtain

$$H = \pi \sum_{i=1}^{N} \sum_{j=1}^{N} c_i c_j \mathcal{M}(\mathbf{x}_i | \mathbf{x}_j),$$

where $\mathcal{M}(\mathbf{x}_i|\mathbf{x}_j)$ is given by (8). One can use the symmetry of the kernel function $\mathcal{M}(\mathbf{x}_i|\mathbf{x}_j)$ to simplify the double sum in this relation. It can be easily shown that

$$\frac{\partial H}{\partial x_i} = c_i \frac{\partial}{\partial x} \frac{\partial H}{\partial \xi}\bigg|_{\mathbf{x}_i} \quad \text{and} \quad \frac{\partial H}{\partial y_i} = c_i \frac{\partial}{\partial y} \frac{\delta H}{\delta \xi}\bigg|_{\mathbf{x}_i}. \tag{A1}$$

Hence using (A1) and $\xi(\mathbf{x})$ the Poisson bracket of ξ and H can be represented by

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$$\{\xi,H\}=\sum_{i=1}^{N}\frac{1}{c_i}\left(\frac{\partial\xi}{\partial x_i}\frac{\partial H}{\partial y_i}-\frac{\partial\xi}{\partial y_i}\frac{\partial H}{\partial x_i}\right).$$

Since ξ obtains its t dependence through $\mathbf{x}_i$ we can substitute ξ and $\{,\}$ in Eq. (10) to yield

$$c_i\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i}\quad\text{and}\quad c_i\frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i}.$$

Note that these equations are similar to Hamilton equations for point vortices in plane.

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