

LAGRANGIAN AVERAGING FOR COMPRESSIBLE FLUIDS*

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Abstract. This paper extends the derivation of the Lagrangian averaged Euler (LAE- α) equations to the case of barotropic compressible flows. The aim of Lagrangian averaging is to regularize the compressible Euler equations by adding dispersion instead of artificial viscosity. Along the way, the derivation of the isotropic and anisotropic LAE- α equations is simplified and clarified.

The derivation in this paper involves averaging over a tube of trajectories η^ϵ centered around a given Lagrangian flow η . With this tube framework, the LAE- α equations are derived by following a simple procedure: start with a given action, Taylor expand in terms of small-scale fluid fluctuations ξ , truncate, average, and then model those terms that are nonlinear functions of ξ . Closure of the equations is provided through the use of *flow rules*, which prescribe the evolution of the fluctuations along the mean flow.

Key words. averaged Lagrangians, inviscid compressible fluids

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1. Introduction.

Historical remarks. The incompressible case will be discussed first. The Lagrangian averaged Euler (LAE- α) equations for average incompressible ideal fluid motion first appeared in the context of averaged fluid models in Holm, Marsden, and Ratiu (1998a, 1998b). Dissipation was added later to produce the Lagrangian averaged Navier–Stokes (LANS- α) equations, also known as the Navier–Stokes- α equations.¹

Remarkably, the LAE- α equations are mathematically identical to the inviscid second grade fluid equations introduced in Rivlin and Erickson (1955), except for the fact that the parameter α is interpreted differently in the two theories. In the case of LAE- α and LANS- α , the parameter α is a spatial scale below which rapid fluctuations are smoothed by linear and nonlinear dispersion.

As in, for example, the work of Whitham (1974) on nonlinear waves, the distinctive feature of the Lagrangian averaging approach is that averaging is carried out at the level of the variational principle and not at the level of the Euler or Navier–Stokes

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¹Sometimes the term “viscous Camassa–Holm (VCH) equations” (Chen et al. (1998)) has been used, but this terminology is a little unfortunate since the n -dimensional version of the CH equations, also known as the EPDiff equations, arise via Euler–Poincaré reduction of H^1 geodesics on the group of all diffeomorphisms and *not* the volume-preserving ones (see Holm and Marsden (2004)).

equations, which is the traditional averaging or filtering approach used for both the Reynolds averaged Navier–Stokes (RANS) and the large eddy simulation (LES) models. As such, the variational procedure does not add any *artificial* viscosity, a physical reason to consider the LAE- α or LANS- α equations as good models for incompressible turbulent flow. Moreover, it has been proven that the α models are computationally very attractive (see Chen et al. (1999) and Mohseni et al. (2003)).

Although sharing the same general technique (use of averaging and asymptotic methods in the variational formulation), several alternative derivations of incompressible LAE- α equations exist in the literature. One of these derivations (see Holm (1999)) uses the generalized Lagrangian mean (GLM) theory developed in Andrews and McIntyre (1978).

An alternative derivation of the incompressible LAE- α and LANS- α equations was given in Marsden and Shkoller (2003) by using an ensemble average over the set of solutions of the Euler equations with initial data in a phase-space ball of radius α while treating the dissipative term via stochastic variations. The derivation also uses a turbulence closure that is based on the Lagrangian fluctuations, namely a generalization of the frozen turbulence hypothesis of Taylor (see Taylor (1938)).

Rigorous analysis aimed at proving global well-posedness and regularity of the three-dimensional isotropic and anisotropic LANS- α equations can be found in, for example, Foias, Holm, and Titi (2002) and Marsden and Shkoller (2001, 2003). However, global existence for the inviscid three-dimensional LAE- α remains an open problem.

From a computational viewpoint, numerical simulations of the α models (see Chen et al. (1999) and Mohseni et al. (2003)) show that the LANS- α equations give computational savings comparable to LES models for forced and decaying turbulent flows in periodic domains. For wall-bounded flows, it is expected that either the anisotropic model or a model with varying α needs to be used; the computational efficacy of these methods on such flows remains to be demonstrated.

As far as the compressible case is concerned, the only reference we know of is Holm (2002a). We shall discuss the relation between the work in this reference and the present paper below.

We refer the interested reader to Marsden and Shkoller (2001, 2003) for a more detailed history of the PDE analysis for LAE- α and LANS- α equations and to Mohseni et al. (2003) for a survey and further references about the numerical aspects of these models.

Motivation. In compressible flows there are two major problems at higher wave numbers, or small scales, that require special attention. These are (a) turbulence for high Reynolds number flows (common with incompressible flows) and (b) strong shocks. In both cases the challenge lies in the appropriate representation of small-scale effects. For turbulence, the energy cascade to smaller scales can be balanced by viscous dissipation, resulting in the viscous regularization of the Euler equations.

Historically, viscous dissipation has been used to regularize shock discontinuities. This includes adding to the Euler equation *nonphysical and artificial viscous terms* and Fourier’s law for heat transfer in the shock region (see, e.g., Liepmann and Roshko (1957) and Shapiro (1953)). This way, the steepening effect of the nonlinear convective term is balanced by dissipation. We believe that Lagrangian averaging is a reasonable alternative way to regularize shock waves. The net effect of Lagrangian averaging is to add dispersion instead of dissipation to the Euler equations; that is, one adds terms that redistribute energy in a nonlinear fashion. In other, rather different situations, the technique of balancing a nonlinear convective term by dispersive mechanisms was used by Lax and Levermore (1983) for the KdV equation and by Kawahara (1970)

and Kakutani and Kawahara (1970) for plasma flows.

The competition between nonlinearity and dispersion has, of course, resulted in remarkable discoveries, the most famous being solitons, localized waves that collide elastically, suffering only a shift in phase. The robustness of solitons in overcoming strong perturbations is largely due to a balance between nonlinearity and linear dispersion. Note that in Lagrangian averaging the energy redistribution mechanism that is introduced is nonlinear and might yield other interesting features that warrant further investigation.

Another feature of the compressible Lagrangian averaged Navier–Stokes- α equations (or CLANS- α equations) is that in turbulent flows with shocks the effect of shocks and turbulence are simultaneously modeled by the same technique, namely the Lagrangian averaging method.

Issues addressed in this paper. In this paper we apply the averaged Lagrangian methodology to derive the isotropic and anisotropic averaged models for compressible Euler equations.

One goal of this paper is to present a clear derivation of the averaged equations. We are particularly interested in separating the two issues of averaging and modeling. In the derivation, a new ensemble averaging technique is proposed and investigated. Instead of taking clouds of initial conditions, as in Marsden and Shkoller (2003), we average over a tube of trajectories η^ϵ centered around a given Lagrangian flow η . The tube is constructed by specifying the Lagrangian fluctuations $\xi^\epsilon = \eta^\epsilon \circ \eta^{-1}$ at $t = 0$ and providing a *flow rule* which evolves them to all later times. The choice of flow rule is a precise modeling assumption which brings about closure of the system.

For the incompressible case we assume that fluctuations are Lie advected by the mean flow (or frozen into the mean flow as divergence-free vector fields), and we obtain both the isotropic and the anisotropic versions of the LAE- α equations. The advection hypothesis is the natural extension to vector fields of the classical frozen turbulence hypothesis of Taylor (see Taylor (1938)) stated for scalar fluctuations.

The second goal of this work is to extend the derivation to barotropic compressible flows. This problem has already been considered by Holm (see Holm (2002a)) in the context of GLM motion. In this work, an alpha model appears as a GLM fluid theory with an appropriate Taylor hypothesis closure. However, even though Holm (2002a) enumerates several frozen-in closure hypotheses, the averaged equations are derived only for the case when the fluctuations are parallel transported by the mean flow. In our work we will consider a more general advection hypothesis to study the compressible anisotropic case. In addition, a physically based new flow rule is introduced to deal with the isotropic case.

The averaging technique consists of expanding the original Lagrangian with respect to a perturbation parameter ϵ , truncating the expansion to $O(\epsilon^2)$ terms, and then taking the average. It turns out that the averaged compressible Lagrangian depends on the Lagrangian fluctuations ξ' only through three tensor quantities which are quadratic in ξ' . In the terminology of Holm (2002a) these tensors represent the second-order statistics of the Lagrangian fluctuations. Evolution equations for these tensors are derived from a core modeling assumption: a prescribed *flow rule* for the time evolution of the fluctuations ξ' . The flow rule gives us closure, allowing us to apply Hamilton's principle to the averaged Lagrangian and thereby derive an equation for the mean velocity u .

The organization of the rest of the paper is as follows. In section 2 we describe a general procedure for Lagrangian ensemble averaging. This procedure is then applied to the action for incompressible fluids in section 3 to demonstrate our derivation

technique. The general procedure is applied again in section 4, this time to the more complex case of barotropic compressible fluids. Section 5 is devoted to modeling issues; here the strategy of modeling the evolution of Lagrangian fluctuations ξ^t using flow rules is discussed in detail. In section 6 we derive the averaged equations for incompressible and compressible models in both isotropic and anisotropic versions. The appendix provides technical details about the fluctuation calculus used throughout the paper.

Main results. The main result of this paper is the derivation of compressible Lagrangian averaged Euler equations with

- anisotropic modeling of fluid fluctuations—see equations (6.2);
- isotropic modeling of fluid fluctuations—see equations (6.4).

In addition, we provide an improved derivation of the *incompressible* isotropic and anisotropic LAE- α equations.

2. General Lagrangian averaging. A mathematical setting for a certain class of compressible fluid flow problems will be given first. After describing the general procedure for Lagrangian averaging, the specific case of the Euler action for fluids will be considered.

Let M be an open subset of \mathbb{R}^N representing the containing space of a fluid. Suppose we are given a Lagrangian for a compressible fluid, $L(\psi, \dot{\psi}, \mu_0)$, where $\psi \in \text{Diff}(M)$, the space of diffeomorphisms of M , $(\psi, \dot{\psi}) \in T\text{Diff}(M)$, and $\mu_0 \in \Lambda^N(M)$, the space of N -forms on M . Fix a time interval $[0, T]$, and let $\mathcal{C}(\text{Diff}(M))$ be the path space of smooth maps from $[0, T]$ into $\text{Diff}(M)$. Then the action $S : \mathcal{C}(\text{Diff}(M)) \times \Lambda^N(M) \rightarrow \mathbb{R}$ is

$$S(\eta, \mu_0) = \int_0^T L(\eta(t), \dot{\eta}(t), \mu_0) dt.$$

We seek an averaged action $S^\alpha(\eta, \mu_0)$, where α is a length scale characterizing the coarseness of the average. Taking η and μ_0 as given, we shall describe how to compute $S^\alpha(\eta, \mu_0)$.

Remark. It is important to emphasize that for both S and S^α , η is merely a test curve. It is *not* an extremal of the action S . We are trying to average the action S itself, not any fluid dynamical PDE or the solutions of such a PDE. Our final product S^α should not depend at all on an initial choice of the test curve η .

Tube initialization. The first step is to take $\xi^\epsilon(x, t)$ to be a family of diffeomorphisms about the identity. That is,

$$\begin{aligned} \text{for each } \epsilon \geq 0, \quad & \xi^\epsilon(\cdot, t) \in \text{Diff}(M) \text{ for all } t, \text{ and} \\ \text{at } \epsilon = 0, \quad & \xi^\epsilon(x, t) = x \text{ for all } x, t. \end{aligned}$$

Define the vector fields ξ' and ξ'' via

$$\xi' = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \xi^\epsilon \quad \text{and} \quad \xi'' = \left. \frac{\partial^2}{\partial \epsilon^2} \right|_{\epsilon=0} \xi^\epsilon.$$

Use ξ^ϵ to construct a tube of material deformation maps that are close to η by letting $\eta^\epsilon(X, t) = \xi^\epsilon(\eta(X, t), t)$, or, written more compactly,

$$(2.1) \quad \eta^\epsilon = \xi^\epsilon \circ \eta.$$

Here, X is a material point in the reference configuration. Define the spatial velocity by $u^\epsilon(x, t) = \dot{\eta}^\epsilon((\eta^\epsilon)^{-1}(x, t), t)$, where η^ϵ is a given material deformation map. Compactly written, this reads as

$$(2.2) \quad u^\epsilon = \dot{\eta}^\epsilon \circ (\eta^\epsilon)^{-1}.$$

The map u^ϵ is a time-dependent vector field on M ; i.e., for each $\epsilon \geq 0$, and for all t , $u^\epsilon(\cdot, t) \in \mathfrak{X}(M)$.

Averaging. The existence of an averaging operation $\langle \cdot \rangle$ will now be postulated. The properties this operation is required to satisfy and an example of such an operation will be given shortly.

Relationship between u^ϵ and u . It is desirable to have the fluctuations ξ^ϵ centered, on average, about the identity: $\langle \xi^\epsilon(x, t) \rangle = x$ for all positions x at all times t . What is actually needed is that for $n \geq 1$,

$$(2.3) \quad \left\langle \frac{\partial^n \xi^\epsilon}{\partial \epsilon^n} \Big|_{\epsilon=0} \right\rangle = 0.$$

In other words, the n th-order fluid fluctuation vector fields should all have mean zero. Restricting the map to be centered about the identity means simply that the average will not be skewed in an arbitrary direction. From (2.2) and (2.3) one can derive

$$(2.4) \quad \langle u^\epsilon \circ \xi^\epsilon(x, t) \rangle = u(x, t).$$

Equation (2.4) shows in which sense the average of u^ϵ is u in a Lagrangian-mean theory defined by $\langle \eta^\epsilon(\cdot, t) \rangle = \eta(\cdot, t)$. This equation is closely connected with the generalized Lagrangian-mean description of Andrews and McIntyre (1978), where the Lagrangian-mean velocity \bar{u}^L and the fluctuating Eulerian velocity u^ξ are related in a similar way.

Density. For the nonaveraged Lagrangian L , μ_0 is a parameter in the sense of Lagrangian semidirect product theory; see Marsden, Ratiu, and Weinstein (1984) and Holm, Marsden, and Ratiu (1998b). The physical interpretation of μ_0 is as follows. Since μ_0 is an N -form on M , it can be written as

$$\mu_0 = \rho_0 dx^1 \wedge \cdots \wedge dx^N,$$

where ρ_0 is a smooth function on M . Now $\rho_0(X)$ is the density of the fluid at the material point X in the reference configuration. This is in contrast to the spatial density $\rho^\epsilon(x, t)$, which gives us the density of the fluid at the spatial point x at time t . Defining

$$(2.5) \quad \mu^\epsilon = \rho^\epsilon dx^1 \wedge \cdots \wedge dx^N,$$

one has the relationship

$$(2.6) \quad (\eta^\epsilon)_* \mu_0 = \mu^\epsilon.$$

Fluctuation calculus. Because u^ϵ and ρ^ϵ will be expanded, the ϵ -derivatives of u^ϵ and ρ^ϵ need to be calculated. First, define

$$(2.7) \quad u' = \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} u^\epsilon \quad \text{and} \quad u'' = \frac{\partial^2}{\partial \epsilon^2} \Big|_{\epsilon=0} u^\epsilon.$$

By differentiating (2.2), one finds expressions for u' and u'' in terms of u , ξ' , and ξ'' . The calculations can be performed intrinsically using Lie derivative formulae—the results, as found in Marsden and Shkoller (2003), are

$$(2.8a) \quad u' = \partial_t \xi' + [u, \xi'],$$

$$(2.8b) \quad u'' = \partial_t \xi'' + [u, \xi''] - 2\nabla u' \cdot \xi' - \nabla \nabla u(\xi', \xi').$$

In these formulas, the bracket $[x, y] = \mathcal{L}_x y$ is the standard Jacobi–Lie bracket of vector fields on M (see, for example, Abraham, Marsden, and Ratiu (1988)). Next, define

$$(2.9) \quad \rho' = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \rho^\epsilon \quad \text{and} \quad \rho'' = \left. \frac{\partial^2}{\partial \epsilon^2} \right|_{\epsilon=0} \rho^\epsilon.$$

One obtains expressions for ρ' and ρ'' in terms of ρ , ξ' , and ξ'' by differentiating (2.6) (see the appendix for the detailed calculations). The results are

$$(2.10a) \quad \rho' = -\operatorname{div}(\rho \xi'),$$

$$(2.10b) \quad \rho'' = \operatorname{div}(\operatorname{div}(\rho \xi' \otimes \xi')) - \operatorname{div}(\rho \xi'').$$

Averaging operation. In the above development, an averaging operation has been implicitly used. The properties it is required to satisfy will now be spelled out. Let $\mathcal{F}(Y)$ mean the space of smooth, real-valued functions on a manifold Y . If Y is infinite-dimensional, then smoothness is understood in the sense of infinite-dimensional calculus with respect to, for example, suitable Sobolev topologies. These infinite-dimensional technicalities will not be required in any detail in this paper, and so may be treated formally.

As before, the set M is the containing space of the fluid, and α is a small positive number. Let \mathfrak{X} be an appropriately chosen space of fields, designed to model “fluid fluctuations,” on M , and consider the space $Y = [0, \alpha] \times \mathfrak{X}$. Assume that there is an averaging operation

$$\langle \cdot \rangle : \mathcal{F}(Y) \rightarrow \mathcal{F}(M)$$

satisfying the following properties for $f, g \in \mathcal{F}(Y)$, $a, b \in \mathbb{R}$, $\psi \in \mathcal{F}([0, \alpha])$, and $h \in \mathcal{F}(\mathfrak{X})$:

$$(2.11) \quad \text{Linearity: } \langle af + bg \rangle = a\langle f \rangle + b\langle g \rangle,$$

$$(2.12) \quad \text{Independence: } \langle \psi h \rangle = \frac{1}{\alpha} \left(\int_0^\alpha \psi(\epsilon) d\epsilon \right) \langle h \rangle,$$

$$(2.13) \quad \text{Commutativity: } \left\langle \int f dx \right\rangle = \int \langle f \rangle dx,$$

$$(2.14) \quad \langle \partial f \rangle = \partial \langle f \rangle, \text{ where } \partial = \partial_t \text{ or } \partial = \partial_{x^i}.$$

Here, $\psi h \in \mathcal{F}(Y)$ is defined as the pointwise product. Note that if ψ is a constant, then the first and second requirements are compatible.

For compressible flow, the space of fluid fluctuations is $\mathfrak{X} = \mathfrak{X}(M)$. For incompressible flow, the space of divergence-free vector fields is used instead, i.e., $\mathfrak{X} = \mathfrak{X}_{\operatorname{div}}(M)$. In general, $\mathfrak{X} = T_{\operatorname{Identity}} X$, where X is the space to which the tube maps ξ^ϵ belong.

Example. Let μ be a probability measure on the unit sphere S in $\mathfrak{X}(M)$, and define the *average* of a (vector-valued) function $f(\epsilon, w)$ on $[0, \alpha] \times S$ by

$$\langle f \rangle := \frac{1}{\alpha} \int_0^\alpha \int_S f(\epsilon, w) d\mu(w) d\epsilon.$$

One checks formally that this is an example of an averaging operation that satisfies the desired properties.

3. Incompressible flow revisited. Before applying the averaging technique to the case of compressible flow, we shall first derive averaged equations for incompressible flow, equations which have already been derived in the literature. The presentation given here has the advantage of being easily generalized to compressible flows. This advantage stems from the careful use and interpretation of modeling assumptions on the fluctuations ξ' —only intuitive assumptions are required regarding the mean behavior of the fluctuations, as well as a first-order Taylor hypothesis. Furthermore, great care has been taken to separate the algebraic issues involved with the averaging procedure from the modeling issues.

In the incompressible case, fluid fluctuations are modeled using the *volume-preserving* diffeomorphism group on M , which is denoted by $\text{Diff}_{\text{vol}}(M)$. Therefore, the tube construction from the previous section now reads as follows: let $\xi^\epsilon(x, t)$ be a family of volume-preserving diffeomorphisms about the identity. That is,

$$\begin{aligned} \text{for each } \epsilon \geq 0, \quad & \xi^\epsilon(\cdot, t) \in \text{Diff}_{\text{vol}}(M) \text{ for all } t, \text{ and} \\ \text{at } \epsilon = 0, \quad & \xi^\epsilon(x, t) = x \text{ for all } x, t. \end{aligned}$$

This forces $\xi'(\cdot, t)$ to be a divergence-free vector field for all t .

Averaged Lagrangian for incompressible fluids. Let us start with the standard Lagrangian

$$(3.1) \quad l(u^\epsilon) = \int_M \frac{1}{2} \|u^\epsilon\|^2 dx$$

and expand u^ϵ in a Taylor series about u :

$$(3.2) \quad u^\epsilon = u + \epsilon u' + \frac{1}{2} \epsilon^2 u'' + \mathcal{O}(\epsilon^3).$$

Substituting this expansion into (3.1) gives

$$(3.3) \quad l(u^\epsilon) = \int_M \frac{1}{2} \|u\|^2 + \epsilon u \cdot u' + \frac{\epsilon^2}{2} (\|u'\|^2 + u'' \cdot u) + \mathcal{O}(\epsilon^3) dx.$$

Let $\hat{l}(u^\epsilon)$ be the truncation of l to terms of order less than ϵ^3 . Using formulas (2.8), u' and u'' can be rewritten in terms of u , ξ' , and ξ'' . We do this in order to write \hat{l} as a function only of u , ξ' , and ξ'' . Making the substitutions and rewriting in coordinates,

$$\begin{aligned} \hat{l}(u^\epsilon) = & \int_M \frac{1}{2} u^i u^i + \epsilon \left(u^i (\partial_t \xi'^i) + u^i u^j \xi'^i_{,j} - u^i \xi'^j u^i_{,j} \right) \\ & + \frac{\epsilon^2}{2} \left((\partial_t \xi'^i) (\partial_t \xi'^i) + 2(\partial_t \xi'^i) \xi'^i_{,k} u^k - 2(\partial_t \xi'^i) u^i_{,k} \xi'^k + \xi'^i_{,j} u^j \xi'^i_{,k} u^k \right. \\ & - \xi'^i_{,j} u^j u^i_{,k} \xi'^k - u^i_{,j} \xi'^j \xi'^i_{,k} u^k + u^i_{,j} \xi'^j u^i_{,k} \xi'^k - 2(\partial_t \xi'^i_{,j}) \xi'^j u^i - 2\xi'^i_{,jk} u^k \xi'^j u^i \\ & \left. - 2\xi'^i_{,k} u^k_{,j} \xi'^j u^i + 2u^i_{,kj} \xi'^k \xi'^j u^i + 2u^i_{,k} \xi'^k_{,j} \xi'^j u^i - u^i_{,jk} \xi'^j \xi'^k u^i \right) \\ (3.4) \quad & + \frac{\epsilon^2}{2} \left((\partial_t \xi''^i) u^i + u^j \xi''^i_{,j} u^i - \xi''^j u^i_{,j} u^i \right) dx, \end{aligned}$$

where the notation $u^i_{,j}$ means $\partial u^i / \partial x^j$. Throughout this paper, there is an implied sum over repeated indices. The averaged Lagrangian for incompressible flow is now simply $l_{\text{in}}^\alpha = \langle \hat{l} \rangle$.

Zero-mean fluctuations. Before undertaking this computation, recall from section 2 that the fluctuation diffeomorphism maps ξ^ϵ are required to have as their average the identity map. This statistical assumption regarding the behavior of the fluctuations is the first modeling assumption:

$$(3.5) \quad \langle \xi' \rangle = 0 \quad \text{and} \quad \langle \xi'' \rangle = 0.$$

This point would not be worth belaboring, except that, when combined with the properties of our averaging operation (2.11)–(2.14), assumption (3.5) forces *all* linear functions of ξ' , ξ'' , and their derivatives to also have zero mean. Applying this fact to (3.4) causes the entire $\mathcal{O}(\epsilon)$ group and the second $\mathcal{O}(\epsilon^2)$ group (i.e., the last line of (3.4)) to vanish inside the average.

We continue analyzing (3.4): the only remaining terms are $(1/2)u^i u^i$ and the first $\mathcal{O}(\epsilon^2)$ group. Within this $\mathcal{O}(\epsilon^2)$ group, we integrate certain terms by parts and notice that all terms involving time derivatives of ξ' group together:

$$(3.6) \quad (\partial_t \xi'^i)(\partial_t \xi'^i) + 2(\partial_t \xi'^i) \xi'^i_{,k} u^k + \xi'^i_{,j} u^j \xi'^i_{,k} u^k \\ = \left((\partial_t \xi'^i) + \xi'^i_{,j} u^j \right) \left((\partial_t \xi'^i) + \xi'^i_{,k} u^k \right) = \left\| \frac{D\xi'}{Dt} \right\|^2,$$

where D/Dt is the material derivative:

$$(3.7) \quad \frac{D}{Dt} = (\partial_t + u \cdot \nabla).$$

We then simplify the remaining non-time-derivative terms from (3.4), integrating by parts to remove second-order spatial derivatives. The final expression for the averaged incompressible Lagrangian is

$$(3.8) \quad l_{\text{in}}^\alpha(u) = \int_M \left\{ \frac{1}{2} \|u\|^2 + \frac{\alpha^2}{2} \left[\left\langle \left\| \frac{D\xi'}{Dt} \right\|^2 \right\rangle - \frac{1}{2} \langle \text{tr}(\nabla \xi' \cdot \nabla \xi') \rangle \|u\|^2 \right] \right\} dx.$$

Modeling of ξ' . Immediate application of Hamilton's principle to (3.8) does not yield a closed system of equations. Namely, we have initial ($t = 0$) data for ξ' but no way to compute this vector field for $t > 0$. Our approach in what follows will be to write down, based on physical considerations, an evolution law, or *flow rule*, for ξ' .

A flow rule consists of a prescribed choice of ϕ in the following evolution equation for ξ' :

$$(3.9) \quad \frac{D\xi'}{Dt} = \phi(u, \rho, \xi').$$

Given a choice of ξ' at $t = 0$, this equation will uniquely determine ξ' for $t > 0$. Let us assume we have a *linear flow rule*,

$$(3.10) \quad \frac{D\xi'^i}{Dt} = \Omega^{ij} \xi'^j,$$

where Ω^{ij} is allowed to depend on u and ρ but not on ξ^ϵ or its derivatives. The caveat here is that our choice of Ω must be compatible with incompressibility; in particular, $\text{div } \xi' = 0$ at $t = 0$, and Ω must be chosen such that ξ' remains divergence-free as it evolves. At this stage, one might raise the issue of the tube ξ^ϵ and request a concrete description of the whole object. Such a description is unnecessary; in order to close the system of evolution equations resulting from (3.8), we need only describe the evolution of the first-order fluctuation field ξ' . Now defining the *Lagrangian covariance tensor*

$$(3.11) \quad F = \langle \xi' \otimes \xi' \rangle$$

and using the linear flow rule (3.10), the Lagrangian (3.8) can be rewritten as

$$(3.12) \quad I_{\text{in}}^\alpha(u) = \int_M \left\{ \frac{1}{2} u^i u^i + \frac{\alpha^2}{2} \left[\Omega^{ij} \Omega^{ik} F^{jk} - \frac{1}{2} F^{ij} u^k u^k \right] \right\} dx.$$

Here we have used the fact that ξ' must be divergence-free.

Advection flow rule. The first flow rule we shall consider results from setting $\Omega^{ij} = u^i_{,j}$:

$$(3.13) \quad \frac{D\xi'^i}{Dt} = u^i_{,j} \xi'^j.$$

Using the definition of the material derivative, it is trivial to see that this flow rule is equivalent to Lie advection of ξ' : $\partial_t \xi' = -\mathcal{L}_u \xi'$. This advection hypothesis is the vector field analogue of the classical frozen turbulence hypothesis of Taylor introduced in Taylor (1938). This hypothesis is widely used in the turbulence community (see Cocke (1969), for instance, for usage of this hypothesis even in the sense of Lie advection of vector fields). More recently, this generalized version of the Taylor hypothesis has been used to achieve turbulence closure in the derivation of incompressible LAE- α equations (see Marsden and Shkoller (2001, 2003)) or in the work of Holm (see Holm (2002a)) on averaged compressible models using the GLM theory.

The advection flow rule (3.13) is perhaps the most obvious choice for Ω that is compatible with incompressibility. Note that if $\text{div } \xi' = 0$ at $t = 0$, then differentiating (3.13) with respect to x^i yields

$$\partial_t (\text{div } \xi') = u^i_{,j} \xi'^j_{,i} - \xi'^i_{,j} u^j_{,i} = 0.$$

Therefore, $\text{div } \xi' = 0$ for all $t > 0$. Using this flow rule, both anisotropic and isotropic models shall be developed. For incompressible flow, no other flow rules will be considered.

Incompressible, anisotropic, inhomogeneous flow. In this case, the flow rule is used to derive an evolution equation for the covariance tensor F . Time-differentiating $F^{ij} = \langle \xi'^i \xi'^j \rangle$ and using (3.13) yields the Lie advection equation $\partial_t F = -\mathcal{L}_u F$. Equipped with an evolution equation for F , we can apply Hamilton's principle to (3.12) and derive a closed system with unknowns u , the average velocity, and F , the covariance tensor.

Carrying this out, one finds that the anisotropic LAE- α equations are given by the following coupled system of equations for u and F :

$$(3.14a) \quad \partial_t (1 - \alpha^2 C)u + (u \cdot \nabla)(1 - \alpha^2 C)u = -\text{grad } p,$$

$$(3.14b) \quad \text{div } u = 0,$$

$$(3.14c) \quad \partial_t F + \nabla F \cdot u - F \cdot \nabla u - \nabla u^T \cdot F = 0,$$

where p is the fluid pressure, and the operator C is defined by

$$(3.15) \quad Cu = \operatorname{div}[\nabla u \cdot F].$$

When $\alpha = 0$, the system (3.14a)–(3.14b) reduces to the incompressible Euler equation.

Note. Start with the generic incompressible averaged Lagrangian (3.12) and substitute the advection flow rule (3.13). Now integrate the last term by parts and use $\operatorname{div} \xi' = 0$. The result is

$$(3.16) \quad l_{\text{in}}^\alpha(u) = \int_M \left\{ \frac{1}{2} \|u\|^2 - \frac{\alpha^2}{2} u \cdot [\nabla \nabla u : F] \right\} dx,$$

which is exactly the Lagrangian used in Marsden and Shkoller (2003) to derive the anisotropic LAE- α equations. However, in Marsden and Shkoller (2003) the second-order Taylor hypothesis

$$\frac{D}{Dt} \langle \xi'' \rangle \perp u,$$

where the orthogonality is taken in L^2 , is necessary to achieve closure. Our choice of modeling assumptions rendered unnecessary any such hypothesis on the second-order fluctuations ξ'' . Second-order Taylor hypotheses, unlike the first-order hypothesis retained from Marsden and Shkoller (2003), do not have much precedent in the turbulence literature, as discussed above.

Incompressible, isotropic, homogeneous fluids. To model the motion of an approximately isotropic fluid, we take the covariance tensor F to be the identity matrix, i.e.,

$$(3.17) \quad F^{ij} = \langle \xi'^i \xi'^j \rangle = \delta^{ij}.$$

The choice of $F^{ij} = \delta^{ij}$ is a modeling assumption, and thus will be valid only for flows which almost preserve this property. Note that (3.17) is strictly inconsistent with the advection flow rule, and thus can be regarded only as an approximation.

For the case of incompressible isotropic mean flow, we assume that (3.17) holds; then differentiating this equation with respect to x^k and x^j and using the fact that ξ' is divergence-free, we have

$$\langle \xi'^i_{,j} \xi'^j_{,k} \rangle = - \langle \xi'^i_{,jk} \xi'^j \rangle.$$

Hence

$$\langle \operatorname{tr}(\nabla \xi' \cdot \nabla \xi') \rangle = \langle \xi'^i_{,j} \xi'^j_{,i} \rangle = - \langle \xi'^i_{,ji} \xi'^j \rangle = 0,$$

and the Lagrangian (3.8) simplifies to

$$(3.18) \quad l_{\text{in,iso}}^\alpha(u) = \int_M \left\{ \frac{1}{2} \|u\|^2 + \frac{\alpha^2}{2} \left\langle \left\| \frac{D\xi'}{Dt} \right\|^2 \right\rangle \right\} dx.$$

We emphasize that this is only an approximation, so that

$$l_{\text{in,iso}}^\alpha(u) \approx l_{\text{in}}^\alpha(u)$$

along fluid trajectories $u(t)$ for which the covariance tensor is approximately the identity. Now using the flow rule given by (3.13), the averaged Lagrangian l_{in}^α from (3.18) becomes

$$(3.19) \quad \left\langle \left\| \frac{D\xi'}{Dt} \right\|^2 \right\rangle = u_{,j}^i u_{,k}^i \langle \xi'^j \xi'^k \rangle = u_{,j}^i u_{,j}^i,$$

where we have used the isotropy assumption (3.17). Hence, (3.18) becomes

$$(3.20) \quad l_{\text{in}}^\alpha(u) = \int_M \left\{ \frac{1}{2} \|u\|^2 + \frac{\alpha^2}{2} \|\nabla u\|^2 \right\} dx.$$

This expression for the averaged Lagrangian in the isotropic case is identical to the one derived in Marsden and Shkoller (2001). Now applying either Hamilton's principle or Euler–Poincaré theory, we obtain the standard isotropic LAE- α equations

$$(3.21a) \quad \partial_t(1 - \alpha^2 \Delta)u + (u \cdot \nabla)(1 - \alpha^2 \Delta)u - \alpha^2 (\nabla u)^T \cdot \Delta u = -\text{grad } p,$$

$$(3.21b) \quad \text{div } u = 0,$$

where p is the usual fluid pressure.

4. Averaged Lagrangian for compressible flow. Having understood the incompressible case, we now turn to the compressible case. The procedure is identical in all aspects, except we must now keep track of density fluctuations. Start with the reduced Lagrangian for compressible flow:

$$(4.1) \quad l(u^\epsilon, \rho^\epsilon) = \int_M \left(\frac{1}{2} \|u^\epsilon\|^2 - W(\rho^\epsilon) \right) \rho^\epsilon dx.$$

The fluid is assumed to be barotropic, meaning that W , the potential energy, is a function only of ρ , the fluid density. Now expand the velocity and density in Taylor series

$$(4.2) \quad \begin{aligned} u^\epsilon &= u + \epsilon u' + \frac{1}{2} \epsilon^2 u'' + \mathcal{O}(\epsilon^3), \\ \rho^\epsilon &= \rho + \epsilon \rho' + \frac{1}{2} \epsilon^2 \rho'' + \mathcal{O}(\epsilon^3) \end{aligned}$$

and also expand the potential energy W :

$$W(\rho^\epsilon) = W(\rho) + \epsilon W'(\rho) \rho' + \frac{1}{2} \epsilon^2 (W''(\rho) \rho'^2 + W'(\rho) \rho'') + \mathcal{O}(\epsilon^3).$$

Substituting these expansions into the reduced Lagrangian gives

$$(4.3) \quad \begin{aligned} l(u^\epsilon, \rho^\epsilon) &= \int_M \left(\frac{1}{2} \|u\|^2 - W(\rho) \right) \rho \\ &\quad + \epsilon \left[(u \cdot u' - W'(\rho) \rho') \rho + \left(\frac{1}{2} \|u\|^2 - W(\rho) \right) \rho' \right] \\ &\quad + \epsilon^2 \left[\frac{1}{2} \left((\|u'\|^2 + u'' \cdot u) - (W''(\rho) \rho'^2 + W'(\rho) \rho'') \right) \rho \right. \\ &\quad \left. + (u \cdot u' - W'(\rho) \rho') \rho' + \frac{1}{2} \left(\frac{1}{2} \|u\|^2 - W(\rho) \right) \rho'' \right] + \mathcal{O}(\epsilon^3) dx. \end{aligned}$$

This expansion is now truncated, leaving out all terms of order ϵ^3 and higher. Denote the truncated Lagrangian by $\hat{l}(u^\epsilon, \rho^\epsilon)$, and define the averaged Lagrangian l^α by

$$(4.4) \quad l^\alpha(u, \rho) = \langle \hat{l}(u^\epsilon, \rho^\epsilon) \rangle.$$

We now outline the procedure by which we arrive at a final written expression for the averaged Lagrangian l^α . The algebra is straightforward but tedious, so details will be omitted.

1. Use (2.8) and (2.10) to rewrite (4.3) in terms of only u , ρ , and the fluctuations ξ' , ξ'' .
2. Remove two kinds of terms that vanish inside the average:
 - (a) linear functions of ξ' or ξ'' ,
 - (b) linear functions of derivatives (either spatial or temporal) of ξ' or ξ'' .
 Note: see “zero-mean fluctuations” in section 3 for justification.
3. Carry out the averaging operation. As in the incompressible case, the only quantities left inside the average should be nonlinear functions of ξ' .

The end result for the averaged Lagrangian for compressible flow is

$$(4.5) \quad l_{\text{comp}}^\alpha(u, \rho) = \int_M \left\{ \frac{1}{2} \rho \|u\|^2 - \rho W(\rho) + \alpha^2 \left[\frac{1}{2} \rho \left\langle \left\| \frac{D\xi'}{Dt} \right\|^2 \right\rangle - \frac{1}{2} w'(\rho) \langle \text{div}(\rho \xi')^2 \rangle - \frac{1}{2} w(\rho) \langle \text{div} \text{div}(\rho \xi' \otimes \xi') \rangle \right] \right\} dx.$$

We have introduced w , the enthalpy,² defined by

$$(4.6) \quad w(\rho) = W(\rho) + \rho W'(\rho).$$

5. Flow rule modeling. In deriving the expressions (4.5) and (3.8) for the averaged Lagrangians, no assumptions were made regarding how the Lagrangian fluctuations ξ' evolve. In this section we describe one possible strategy for modeling ξ' . Note that such a strategy is necessary to achieve closure for the evolution equations associated with the Lagrangians (4.5) or (3.8).

Preliminary observation. Assuming ξ' evolves via a linear flow rule, as in (3.10), the vector field ξ' appears in the averaged Lagrangian (4.5) only as part of the following three expressions:³

$$(5.1a) \quad F^{ij} = \langle \xi'^i \xi'^j \rangle,$$

$$(5.1b) \quad G^i = \langle \xi'^i \xi'^j_{,j} \rangle,$$

$$(5.1c) \quad H = \langle \xi'^i_{,i} \xi'^j_{,j} \rangle.$$

Note that F is the same *Lagrangian covariance tensor* from the incompressible derivation. In terms of these quantities, the averaged compressible Lagrangian is given in

²Any function w satisfying $\nabla w = (\nabla p)/\rho$, where p is pressure, is called enthalpy. Our definition of w implies $w_{,i} = 2W'(\rho)\rho_{,i} + \rho W''(\rho)\rho_{,i} = (\rho^2 W'(\rho))_{,i}/\rho = p_{,i}/\rho$, as required.

³Similar tensors appear in Holm (1999); they are referred to as second-order statistics of the Lagrangian fluctuations.

coordinates by

(5.2)

$$l_{\text{comp}}^\alpha(u, \rho) = \int_M \left\{ \frac{1}{2} \rho u^i u^i - \rho W(\rho) + \alpha^2 \left[\frac{1}{2} \rho \Omega^{ij} \Omega^{ik} F^{jk} - \frac{1}{2} w'(\rho) (\rho_{,i} \rho_{,j} F^{ij} + 2 \rho \rho_{,j} G^j + \rho^2 H) - \frac{1}{2} w(\rho) (\rho F^{ij})_{,ij} \right] \right\} dx.$$

Time-differentiating (5.1a)–(5.1c) and using the linear flow rule (3.10) results in evolution equations for F , G , and H :

$$(5.3a) \quad \partial_t F^{ij} = \Omega^{ik} F^{kj} + \Omega^{jk} F^{ki} - u^k F_{,k}^{ij},$$

$$(5.3b) \quad \partial_t G^i = \Omega^{ik} G^k - u^k G_{,k}^i + F^{ij} \Omega_{,k}^{kj} + \langle \xi'^i \xi'^j_{,k} \rangle (\Omega^{kj} - u_{,j}^k),$$

$$(5.3c) \quad \partial_t H = 2 \Omega_{,i}^{ik} G^k - u^k H_{,k} + 2 \langle \xi'^j_{,k} \xi'^i_{,i} \rangle (\Omega^{kj} - u_{,j}^k).$$

Flow rules. For compressible flows, two flow rules will be considered. We define them first and then go on to consider their relative merits and demerits:

(I) Advection: $\Omega^{ij} = u_{,j}^i$.

(II) Rotation: $\Omega^{ij} = \frac{1}{2}(u_{,j}^i - u_{,i}^j)$.

Advection. For our anisotropic model, we shall advect ξ' and treat the quantities F , G , and H as parameters in the final system, each of which will have its own evolution equation. Substituting $\Omega^{ij} = u_{,j}^i$ into the system (5.3) gives

$$(5.4a) \quad \partial_t F = -\mathcal{L}_u F,$$

$$(5.4b) \quad \partial_t G = -\mathcal{L}_u G + F \cdot \text{grad}(\text{div } u),$$

$$(5.4c) \quad \partial_t H = 2 \text{grad}(\text{div } u) \cdot G - u \cdot \text{grad } H.$$

One advantage of the advection flow rule is that it automatically closes the system (5.3). For a general choice of Ω , the system involves $\langle \xi'^i \xi'^j_{,k} \rangle$ and $\langle \xi'^j_{,k} \xi'^i_{,i} \rangle$, which cannot be expressed solely in terms of F , G , and H .

Rotation. For our isotropic model, we want to know whether the evolution equation (5.3a) for F preserves the isotropy relationship $F = \text{Identity}$. Suppose $F^{ij} = \delta^{ij}$ at $t = 0$. Then substituting into (5.3a) reveals that

$$(5.5) \quad \partial_t|_{t=0} F^{ij} = \Omega^{ij} + \Omega^{ji}.$$

If Ω is antisymmetric, we have $\partial_t|_{t=0} F = 0$, and $F(x, t) = \text{Identity}$ solves (5.3a) for all t . We wish to know whether this solution is unique. This is guaranteed by a straightforward generalization of the results concerning linear hyperbolic systems of first-order equations from Evans (1998), assuming sufficient smoothness of u .

We conclude that antisymmetry of Ω is sufficient to guarantee that the initial data $F = \text{Identity}$ is, in fact, preserved for all t . Then an immediate choice of a tensor Ω that is antisymmetric is given by the rotation flow rule (II). This form has a very attractive physical interpretation. Putting the linear flow rule equation (3.10) together with (II) gives us

$$(5.6) \quad \frac{D\xi'}{Dt} = \omega \times \xi',$$

where $\omega = \text{curl } u$ is the vorticity vector. The last equation can be interpreted in the sense that fluctuations are rigidly transported by the mean flow, with a local angular velocity given by the vorticity vector.

Finally, the rotation flow rule (II) does not by itself close the system (5.3). When using this flow rule, we shall assume that $G = 0$ and $H = \beta^2$.

6. Equations for averaged dynamics. Here we shall write down two systems of coupled PDEs which describe the evolution of the average velocity and density in a compressible flow. Each PDE is derived from an associated averaged Lagrangian.

Compressible, fully anisotropic, inhomogeneous fluids. By substituting (I) into the Lagrangian (4.5), we obtain closure: the Lagrangian no longer depends explicitly on ξ' , instead depending on the tensors F , G , and H , for which a self-contained system of evolution equations (5.4) has already been derived—see section 5 for details. Applying Hamilton's principle directly to (5.2) yields an evolution equation for u , the average fluid velocity. We write this equation using the operator \mathcal{A} , which is defined as

$$(6.1) \quad (\mathcal{A}v)^i = \frac{1}{\rho} (\rho v_{,j}^i F^{jk})_{,k}.$$

We also write $\tilde{w} = \rho w'(\rho)$, where $'$ means $d/d\rho$ as usual. The anisotropic compressible LAE- α equations are

$$(6.2a) \quad (\partial_t u^n + u_{,i}^n u^i) = (1 - \alpha^2 \mathcal{A})^{-1} \frac{1}{\rho} \left\{ -\rho w_{,n} - \frac{\alpha^2}{2} \left[\rho (F^{ij} u_{,i}^k u_{,j}^k)_{,n} + F_{,ij}^{ij} \rho \tilde{w}_{,n} \right. \right. \\ \left. \left. + F_{,n}^{ij} \rho_{,i} \tilde{w}_{,j} + (F_{,n}^{ij} \rho)_{,ij} \tilde{w} + 2G_{,n}^i \rho \tilde{w}_{,i} + 2G^i (\rho \tilde{w}_{,n})_{,i} + (H \rho^2 \tilde{w}')_{,n} \right] \right\},$$

$$(6.2b) \quad \partial_t \rho = -\text{div}(\rho u),$$

$$(6.2c) \quad \partial_t F = -\nabla F \cdot u + F \cdot \nabla u + \nabla u^T \cdot F,$$

$$(6.2d) \quad \partial_t G = -u \cdot \nabla G + G \cdot \nabla u + F \cdot \text{grad}(\text{div } u),$$

$$(6.2e) \quad \partial_t H = 2 \text{grad}(\text{div } u) \cdot G - u \cdot \text{grad } H.$$

Well-posedness. We now sketch a rough well-posedness argument for the system (6.2). Assume that the tensor F is positive definite. By this it is meant, since F is a $(2,0)$ tensor, that for any one-form θ , the contraction $F : (\theta \otimes \theta)$ is positive everywhere. Given the ρ -weighted inner product $\langle f, g \rangle = \int f g \rho$, we have $\langle f, -\mathcal{A}f \rangle = -\int f \frac{1}{\rho} (\rho f_{,j} F^{jk})_{,k} \rho = \int f_{,j} F^{jk} f_{,k} \rho > 0$. Since $-\mathcal{A}$ is a positive definite linear operator, the kernel of $(1 - \alpha^2 \mathcal{A})$ is trivial, and we expect that (6.2) is well-posed.

It would be of analytical interest to see to what extent the “geodesic part” of these equations defines a smooth spray in the sense of Ebin and Marsden (1970), and which holds for the EPDiff equations (that is, the n -dimensional CH equations), as explained in Holm and Marsden (2004).

Compressible, isotropic, inhomogeneous. For this case we use flow rule (II), which can be written in vector notation as

$$\Omega = \frac{1}{2} (\nabla u - \nabla u^T).$$

Recall that this flow rule is compatible with an isotropic choice of the covariance tensor, i.e., $F^{ij} = \delta^{ij}$. We further assume that $G = 0$ and $H = \beta^2$ for some constant β .

Using flow rule (II) along with these extra assumptions in the general Lagrangian expression (5.2) gives us a Lagrangian in only two variables:

$$(6.3) \quad l(u, \rho) = \int_M \left(\frac{1}{2} \rho \|u\|^2 - \rho W(\rho) + \alpha^2 \left[\frac{1}{4} \rho (\|\nabla u\|^2 - \text{tr}(\nabla u \cdot \nabla u)) - \frac{1}{2} w'(\rho) (\|\nabla \rho\|^2 + \rho^2 \beta^2) - \frac{1}{2} w(\rho) \Delta \rho \right] \right) d^N x,$$

where $w(\rho) = W(\rho) + \rho W'(\rho)$ is the enthalpy introduced in (4.6). Regarding this as a Lagrangian in u and $\mu = \rho d^N x$, one uses the semidirect product Euler–Poincaré equations (see Holm, Marsden, and Ratiu (1998b)) to derive the system

$$(6.4a) \quad \partial_t(\rho v) + (u \cdot \nabla)(\rho v) + \alpha^2 \text{div}(\rho \Omega \cdot \nabla u) + \rho v \text{div} u = -\nabla \tilde{p},$$

$$(6.4b) \quad \partial_t \rho + \text{div}(\rho u) = 0,$$

with the modified momentum ρv and modified pressure \tilde{p} given by

$$(6.5) \quad \rho v = \rho u + \alpha^2 \text{div}(\rho \Omega),$$

$$(6.6) \quad \nabla \tilde{p} = \nabla p + \alpha^2 \beta^2 \rho \nabla \left(\rho w' + \frac{1}{2} \rho^2 w'' \right).$$

Here are explicit coordinate expressions for two slightly complicated objects:

$$\rho v^i = \rho u^i + \frac{1}{2} \alpha^2 \left(\rho (u^j_{,i} - u^i_{,j}) \right)_{,j},$$

$$\text{div}(\rho \Omega \cdot \nabla u) = (\rho \Omega^{ki} u^i_{,j})_{,k}.$$

The following convention for divergences of tensors has been used: given a 2-tensor A^{ij} , we set

$$(\text{div} A)^j = A^i_{,i}{}^j.$$

That is, the contraction implicit in the divergence operation always takes place on the *first* index.

Observations.

- In the case of homogeneous incompressible flow, where ρ is constant and $\text{div} u = 0$, the definition of ρv in (6.5) reduces to

$$v = \left(1 - \frac{1}{2} \alpha^2 \Delta \right) u,$$

which after rescaling α to get rid of the factor of 1/2 is precisely the v one finds in treatments of the incompressible LAE- α and LANS- α equations.

- The above does not work in one spatial dimension. The problem is that here Ω reduces to $(u_x - u_x)/2 = 0$, which clearly does not describe transport at all. For a one-dimensional isotropic model one may very well want to forget about antisymmetry of Ω and instead use something such as the advection flow rule. One may, quite reasonably, conclude that the only meaning of isotropy in one dimension should be reflection symmetry.

7. Future directions.

The initialization problem. Perhaps the largest unsolved problem for the Lagrangian averaged equations is the initialization problem. A concise statement of the problem reads as follows:

Given initial data $u_0(x)$ for the Euler equation, how does one obtain initial data $U_0(x)$ for the LAE- α equation?

Let us look at this problem in slightly more detail. Let u denote the solution of the incompressible Euler equations for initial data u_0 , i.e., $u(x, 0) = u_0(x)$. Similarly, let U denote the solution of the incompressible, isotropic LAE- α equations (3.21) for initial data U_0 .

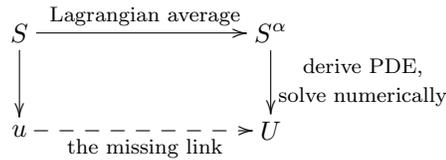
Now U should be, in some sense, the mean flow of the fluid. This means that U_0 should be the mean flow of the fluid at time $t = 0$, implying that U_0 should be, in some sense, an “averaged” or “filtered” version of u_0 . The question is, How does one derive U_0 from u_0 ? Another way of phrasing this question is, How do we describe (approximately) the initial state of the fluid (given exactly, for our purposes, by the field u_0) using only the mean flow variable U_0 ?

Numerous methods have been used to initialize the LAE- α equations for use in numerical simulations, but none of these methods has any theoretical foundation. There is also no theory regarding how one should filter a full Euler flow u , or even a family of flows u^ϵ , in order to obtain a mean flow that could be compared with the full LAE- α trajectory U . In this respect, (2.4), which states that

$$\langle u^\epsilon \circ \xi^\epsilon(x, t) \rangle = u(x, t),$$

is not helpful: we have no way to compute the fluctuation diffeomorphism group ξ^ϵ . Therefore, we have no way to compute the left-hand side $\langle u^\epsilon \circ \xi^\epsilon \rangle$.

The difficulty can be summarized in the following commutative diagram. Here S is the standard Euler action and S^α is the Lagrangian averaged action.



Solid arrows represent steps that we know how to carry out. The dashed arrow represents the one step that we do not know how to carry out. Our strategy for this problem will be to develop methods by which we can test different filters for obtaining U_0 from u_0 in practice.

Treatment of densities. Another area for further investigation involves our treatment of the density tube μ^ϵ . There are two questions to ground us:

1. We have tacitly assumed that at $t = 0$, for all ϵ and all x ,

$$\mu(x, t) = \mu^\epsilon(x, t).$$

An argument similar to the one made above in our discussion of the initialization problem can be made here. Namely, $\mu(x, 0)$ represents the mean density at time $t = 0$. Meanwhile, $\mu^\epsilon(x, 0)$ represents the true density of the fluid at time $t = 0$. These two quantities need not be equal. This prompts the following question: how would we carry out the procedure from sections 2 and 4 with tubes in which each trajectory does *not* have the same initial density $\mu(x, 0)$?

2. As our derivation of the averaged compressible equations stand, we have derived the fact that the “mean” density μ was advected by the mean flow U : $\partial_t \mu = -\mathcal{L}_U \mu$. Substituting $\mu = \rho d^N x$ and using the definition of divergence yields the standard continuity equation

$$\partial_t \rho + \operatorname{div}(\rho U) = 0.$$

In both RANS and LES treatments of averaged/filtered flow, the mean flow U satisfies a *modified* continuity equation rather than the standard one. Therefore, why does the Lagrangian averaged mean density μ satisfy the usual continuity equation?

The two questions regarding densities are, in fact, related. To see this, let us suppose that, given the initial density μ_0 associated with the center line of our tube η , we have a method for constructing a family of initial densities μ_0^ϵ for each of the other curves in the tube η^ϵ . Now defining⁴

$$\mu^\epsilon(t) = (\eta_t^\epsilon)_* \mu_0^\epsilon \quad \text{and} \quad \bar{\mu}(t) = \langle \mu^\epsilon(t) \rangle,$$

we will find that $\bar{\mu}(t) = \bar{\rho}(t) d^N x$ satisfies a modified continuity equation

$$\partial_t \bar{\rho}(t) + \operatorname{div}(\bar{\rho} u) + \operatorname{div} \left\langle \rho^\epsilon \left(\epsilon u' + \frac{1}{2} \epsilon^2 u'' \right) \right\rangle = 0.$$

To close this equation, we either must carry out the average directly or must expand ρ^ϵ about a suitable trajectory and make modeling assumptions.

Filtered Lagrangians. We have seen that the current averaging procedure leads to complicated averaged equations. Furthermore, there is no clear way to evaluate numerically the flow rules we have proposed on physical grounds. One of our immediate goals is to investigate a filtering approach, still at the level of the Lagrangian, which will lead to simpler averaged models that can be tested numerically. The filtering approach we have in mind begins with a decomposition of the velocity field

$$(7.1) \quad u = \bar{u} + u' \quad \text{and} \quad \rho = \bar{\rho} + \rho'$$

into mean and fluctuating components. This would replace the Taylor expansion (4.2) of u^ϵ and ρ^ϵ that we carried out in the present work and would therefore lead to Lagrangians and equations with much less algebraic complexity. As opposed to the axiomatic averaging operation $\langle \cdot \rangle$, the filter shall be specified concretely. We expect this to help greatly with the initialization and density problems discussed above; furthermore, the filtering approach leads naturally to questions about the relationship between LES and LAE- α models.

Simpler models. As we previously noted, the flow rule approach developed in this paper does not yield a one-dimensional compressible averaged model. We are currently investigating such a model, derived from the filtered Lagrangian

$$(7.2) \quad l(\rho, u) = \int \left(\frac{1}{2} uv - W(\rho) \right) \rho d^N x,$$

where $v = (1 - \alpha^2 \partial_{xx})u$. To derive this Lagrangian, we filter only the velocity, leaving density and potential energy alone. This is the compressible analogue of the filtered

⁴Note that $\langle \mu^\epsilon(x, t) \rangle \neq \mu(x, t)$.

Lagrangian used in deriving the CH equation (see Camassa and Holm (1993)). The analysis and numerical simulation of the new equations presented in section 6 of this work will be difficult. The analysis of the PDE associated with (7.2) is much easier. In particular, we expect that numerical studies of this one-dimensional model will yield insight into the dynamics of the higher-dimensional equations.

Entropy. In the derivation of our compressible averaged models, we have made the barotropic assumption $W = W(\rho)$. We expect the resulting barotropic model to be useful in computing mean flow quantities in regimes where we are not concerned with strong physical shocks, for example in climate models. The next major step forward will be to remove the barotropic assumption and derive a model that is valid in regimes where we are concerned with shocks.

To this end, we have derived an averaged model for the general case, where the potential energy has the form $W(\rho, S)$, where S is the entropy. This model, which consists of a system of equations for ρ , u , and S , also involves the pressure p . Therefore, in order to close the system, we require an equation of state relating p to ρ and S . The open question now is as follows: given an equation of state for the compressible Euler system, what is the equation of state relating the averaged variables to one another? In other words, how does Lagrangian averaging interact with the thermodynamics of the system? We hope that analyzing a finite-dimensional case of this interaction will shed light on this issue.

Connections with Kevrekidis's coarse/fine methods. Given a description of any mechanical system, not necessarily involving fluids, in the form of a Lagrangian ℓ , we can carry out the procedure described in section 2 to find an averaged Lagrangian $\langle \ell \rangle$. From this we can derive equations of motion for the average dynamics of the original system. Changing our language slightly, we say that we have a general method for extracting the “coarse” dynamics of a mechanical system whose full description involves motions on both fine and coarse scales.

Another method for computing the coarse-scale dynamics of a mechanical system has been put forth in Kevrekidis et al. (2002). Kevrekidis's method does not involve trying to write down equations of motion which govern the coarse dynamics. Instead, he offers an algorithmic approach, the crux of which is as follows. The coarse dynamics of a system are found by *lifting* the initial ($t = t_0$) state to an ensemble of initial states, *integrating* each using the full equations until some small final time $t = \epsilon$ has been reached, and *projecting* the resulting $t = \epsilon$ states onto a single state. This $t = \epsilon$ state is then *extrapolated* to a state at some desired $t = t_f > 0$. By iterating this process and tuning the lifting, projection, and extrapolation operations, this method can be used to recover the coarse dynamics of the system.

Now the question that begs to be asked is as follows: for the case of fluid dynamics, how different are the coarse dynamics provided by the LANS- α equation from the coarse dynamics one would obtain by following Kevrekidis? The difficulty in answering this question lies in implementing a full fine-scale integrator for fluids that one could successfully embed inside Kevrekidis's coarse-scale algorithm. We look forward to tackling this task soon.

8. Appendix: Fluctuation calculus details. Before proceeding with any derivations, we state the Lie derivative theorem for time-dependent vector fields: if the vector field X_λ has flow F_λ , then

$$(8.1) \quad \frac{d}{d\lambda} F_\lambda^* Y_\lambda = F_\lambda^* \left(\frac{\partial Y_\lambda}{\partial \lambda} + \mathcal{L}_{X_\lambda} Y_\lambda \right).$$

Our task now is to derive equations (2.10). Starting with (2.6), let us move η^ϵ to the right-hand side of the equation:

$$(8.2) \quad \mu_0 = (\eta^\epsilon)^* \mu^\epsilon.$$

The strategy is to differentiate with respect to ϵ and use the Lie derivative theorem (8.1). The intrinsic definition of divergence

$$(8.3) \quad \mathcal{L}_\zeta(\nu) = (\operatorname{div}_\nu \zeta) \nu$$

and the canonical volume form $d^N x = dx^1 \wedge \cdots \wedge dx^N$ will both be used in what follows. Note that $\operatorname{div} \zeta$ with no subscript on the div means $\mathcal{L}_\zeta(d^N x)$. Before applying the Lie derivative theorem, note that the vector field

$$(8.4) \quad W^\epsilon = \frac{\partial}{\partial \epsilon} \eta^\epsilon \circ (\eta^\epsilon)^{-1}$$

has flow η^ϵ . A simple computation yields

$$(8.5) \quad \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} W^\epsilon = \xi'' - \nabla \xi' \cdot \xi'.$$

Then we start with ρ' :

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \mu_0 = 0 &= \frac{\partial}{\partial \epsilon} (\eta^\epsilon)^* \mu^\epsilon && \text{by differentiating (8.2)} \\ &= (\eta^\epsilon)^* \left(\frac{\partial \mu^\epsilon}{\partial \epsilon} + \mathcal{L}_{W^\epsilon} \mu^\epsilon \right) && \text{by (8.1)} \\ &= \eta^* (\mu' + \mathcal{L}_{\xi'} \mu) && \text{at } \epsilon = 0 \\ \implies \mu' &= -\mathcal{L}_{\xi'} \mu, \\ \rho' d^N x &= -(\mathcal{L}_{\xi'} \rho) d^N x - \rho (\mathcal{L}_{\xi'} d^N x) && \text{by (2.5),} \\ \rho' d^N x &= -(\nabla \rho \cdot \xi' + \rho \operatorname{div} \xi') d^N x && \text{by (8.3)} \\ \implies \rho' &= -\operatorname{div}(\rho \xi'). \end{aligned}$$

Next we compute ρ'' :

$$\begin{aligned} \frac{\partial^2}{\partial \epsilon^2} \mu_0 = 0 &= \frac{\partial^2}{\partial \epsilon^2} (\eta^\epsilon)^* \mu^\epsilon \\ &= (\eta^\epsilon)^* \left(\frac{\partial^2 \mu^\epsilon}{\partial \epsilon^2} + \mathcal{L}_{W^\epsilon} \frac{\partial \mu^\epsilon}{\partial \epsilon} + \frac{\partial}{\partial \epsilon} (\mathcal{L}_{W^\epsilon} \mu^\epsilon) + \mathcal{L}_{W^\epsilon} \mathcal{L}_{W^\epsilon} \mu^\epsilon \right) \\ \implies 0 &= \eta^* (\mu'' + 2\mathcal{L}_{\xi'} \mu' + \mathcal{L}_{\xi'' - \nabla \xi' \cdot \xi'} \mu + \mathcal{L}_{\xi'} \mathcal{L}_{\xi'} \mu) \\ \implies \mu'' &= -\mathcal{L}_{\xi''} \mu + 2\mathcal{L}_{\xi'} \mathcal{L}_{\xi'} \mu - \mathcal{L}_{\xi'} \mathcal{L}_{\xi'} \mu + \mathcal{L}_{\nabla \xi' \cdot \xi'} \mu, \\ \rho'' d^N x &= -(\operatorname{div}(\rho \xi'')) d^N x + \operatorname{div}(\operatorname{div}(\rho \xi') \xi') d^N x + \operatorname{div}(\rho \nabla \xi' \cdot \xi') d^N x \\ \implies \rho'' &= -(\operatorname{div}(\rho \xi'')) + \left(\left(\rho \xi'^i \right)_{,i} \xi'^j \right)_{,j} + \left(\rho \xi'^j \right)_{,i} \xi'^i \\ &= -(\operatorname{div}(\rho \xi'')) + \left(\rho \xi'^i \xi'^j \right)_{,ij} \\ &= -(\operatorname{div}(\rho \xi'')) + \operatorname{div} \operatorname{div}(\rho \xi' \otimes \xi'). \end{aligned}$$

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