Attitude Control of Micro/Mini Aerial Vehicles and Estimation of Aerodynamic Angles Formulated as Parametric Uncertainties

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Abstract—A recurring problem in controlling the attitude of miniature/micro aerial vehicles (MAVs) is the presence of an unknown atmospheric wind. Wind directly affects the vehicle's translational variables, namely the airspeed, angle-of-attack, and sideslip angle, all three of which are important variables in the equations that govern a fixed wing aircraft's aerodynamics. Consequently, wind exerts a major influence on the overall dynamics of an MAV. In this letter, we assume a knowledge of airspeed and describe a Lyapunov-based nonlinear estimation and control method that estimates the effects of angle-of-attack and sideslip on the vehicle's attitude, and at the same time controls the attitude to track a desired quaternion trajectory. This makes it possible to design an attitude control module that does not rely upon a knowledge of the individual components of the vehicle's translational velocity. The decomposition of the measured airspeed into the individual components is accomplished by the controller itself, in the form of an estimation of the angle-of-attack and the sideslip angle. We prove that, under quite general conditions, we can achieve asymptotic tracking of the attitude to the desired trajectory and also estimate how the attitude dynamics are affected by unpredictable disturbances in the angle-of-attack and sideslip angle.

Index Terms—Adaptive control of robotic systems, formal methods in robotics and automation, motion control, dynamics.

I. Introduction

TRADITIONAL aircraft control treats the lateral and longitudinal dynamics as primarily decoupled linear perturbed systems. This is a reasonable treatment when we are considering a large aircraft flying at high speeds relative to the ambient wind [1]. The decoupling naturally materializes as insignificant cross derivatives (e.g. $C_{l,\alpha}$, $C_{m,\beta} \ll 1$), while the deviation from trimmed flight manifest as small perturbations in the angular velocities, the angle-of-attack α, and the sideslip-angle β.

For smaller vehicles flying at slower airspeeds, both the linear perturbation and decoupled dynamics assumptions break down. The higher ratio of the ambient wind V_w to the relative airspeed V_a increases the perturbations in the aerodynamic angles α and β which affect all the aerodynamic forces and moments ([2] chapter 4). Moreover, recent work from our group shows that the lateral and longitudinal dynamics can no longer be considered decoupled [3], [1] because of the complicated interplay between tip vortices and the leading-edge separation bubble. Furthermore, the angular velocities and accelerations scale higher as the length scale L is lowered. Applying constant Froude number scaling under the constant external gravitational field, the mass varies as L^3, the moment of inertia as L^5, relative airspeed as $\sqrt{L}$ (so that the aerodynamic forces vary as L^3 to balance the gravitational force), time as $\sqrt{L}$, linear accelerations as 1, the aerodynamic moments (force times moment arm) as L^4, the angular velocities as $1/\sqrt{L}$, and the angular accelerations as $1/L$. This argument assumes the aspect ratio, material density, and the angle-of-attack remain the same as the length is scaled. Despite the deviations from this assumption, the scaling argument that airspeed scales as $\sqrt{L}$ is known to hold to a good accuracy in practice. For e.g., the Aerosonde, which is about a 30th the size of a Boeing 747, flies at nearly a seventh of its airspeed. From the controller's perspective, the implication is that control of the angular variables must be given primary importance in smaller aircraft, more so than the translational control.

In order to address the increased significance of the angular variables in a MAV compared with a larger and faster aircraft, we propose to first control the attitude, and then add an outer loop to track a desired path or trajectory. While doing this, we need to be mindful of the fact that the translational velocities do enter the attitude dynamics in the form of α and β. The problem of controlling the attitude of a MAV is therefore associated with a dual problem of estimation on account of the presence of significant sideslip and angle-of-attack.

The approach presented in this letter is to design a basic nonlinear controller for the attitude of the vehicle, that simultaneously estimates the effect of α and β on the attitude dynamics in the presence of wind. This avoids having to install expensive multiple axes airspeed sensors on the vehicle, or any other kind of additional hardware dedicated to measuring or estimating α and β. We could thus state the problem as one of *controlling the attitude of a fixed-wing MAV in the presence of atmospheric wind, using a knowledge of the attitude, the angular velocity, and the relative airspeed*. The components of the ground-referenced translational velocity are not required for the attitude control
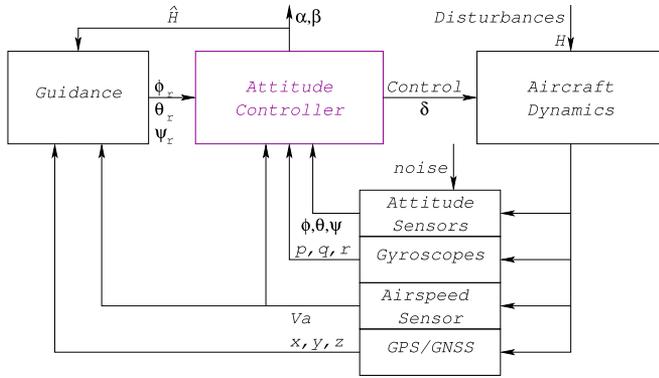
Fig. 1.   The proposed control topology for a MAV.

presented in this letter as such, but are presumably required for the outer-loop that attends to translational guidance and navigation. A schematic of the control topology is shown in Fig. 1.

The attitude state could be estimated using a magnetometer sensor along with a gravity-vector sensor or a zenith-vector sensor. There are several methods to estimate a vehicle's attitude from multiple vector observations: Davenport's q-method [4], an SVD method, the Quaternion Estimator QUEST, the TRIAD [5] etc. A survey of the established methods was provided at the turn of the century by Markley and Mortari in [6]. Three-axis angular velocity measurements can be obtained from electronic gyroscopes in several commercially available inertial measurement unit(IMU)s. Such IMUs typically also contain three-axis accelerometer sensors which provide a measurement of the vehicle's translational acceleration. A working autopilot has been reported by our team [7] that uses the commercial IMU, MPU9250, to estimate its attitude state and the angular velocity. The MPU9250 provides three-axis magnetometer, gyroscope, and accelerometer measurements. For our purposes, we assume the availability of such an IMU, along with the software required to estimate the attitude state, and the angular velocities, and supplement the aerodynamic angle estimation.

The aerodynamic angles can be estimated in the absence of wind by designing an observer around the vehicle accelerations. However measurement of acclerations is known to have significant noise. A typical low-cost accelerometer could have a static 3-sigma noise as high as 2.4% the acceleration due to gravity (e.g., the MPU9250 [8]). This theoretical noise would add over the other practical – and typically much larger – errors associated with the observer design, like the gain-error and distortion introduced by the accelerometers (0.5% and 2% respectively for the MPU9250), the vibrations introduced from a running propeller, crosstalk across the three spatial axes, signal leakage from the gyroscopes to the accelerometers etc. A good design for estimating the effect of the aerodynamic angles would therefore avoid using the acceleration measurements, or atleast carefully filter out the noise in the measurements.

The problem of estimating the aerodynamic angles without using acceleration measurements has already been treated in the literature in various contexts. For example, in [9], Popowski and Dabrowski show how the angles may be determined from a measurement of the translational relative velocity components

of the vehicle. This of course requires installing expensive sensors on the vehicle, or using an unreliable, noisy and lagging GNSS (Global Navigation Satellite System) measurement. To obtain a sense of the significance of the noise and latency in the GNSS measurement, it must be noted that the positional accuracy of a GNSS measurement is around 3 m (civilian channel), and for a vehicle speed of 20 m/s, meaningful (20% accurate) estimates of the velocity can be obtained only at a latency of more than a second. Meanwhile, the aerodynamic angles can vary significantly in that time on account of gusts.

In [10], Oosterom and Babuska use a Neural-Networks based virtual sensor which takes the measured α from physical sensors to either reduce the estimation error or add redundancy. In [11], α and β are estimated by (a) assuming that their rate of change is approximated by the angular velocity of the vehicle and (b) using an extended Kalman Filter on the measured angular velocities and V_a. However, this approach assumes that α and β are solely determined by the vehicle dynamics, and does not account for the important effect of an uncertain wind.

We now briefly lay out the outline of this manuscript. In Section II, we describe the dynamics of the MAV attitude using the quaternion formulation. In the following Section III, we describe the estimation of the effect of α and β on the attitude dynamics while simultaneously controlling the attitude. Finally, Section IV shows simulation results that validate the claims made by the theory in the presence of modelling nonidealities and disturbance uncertainties.

II. ATTITUDE DYNAMICS

We consider the attitude dynamics of a fixed-wing aircraft. Let the relative velocity vector of the aircraft be denoted by $\mathbf{V_a}$. The angle made by $\mathbf{V_a}$ to the aircraft's plane of symmetry, the xz-plane, is the sideslip angle β. The angle made by the projection of $\mathbf{V_a}$ on the xz-plane to the vehicle's longitudinal axis is the angle-of-attack α. The magnitude of the relative velocity vector is denoted as V_a. The flight angle γ is the angle made by $\mathbf{V_a}$ to the local horizontal plane.

As a representative MAV, we use the Aerosonde model provided in [2]. The algorithmic development is, however, independent of the exact values of the model parameters, and can be easily rendered to work with the model of a different type of airplane. Moreover, the controller can also accommodate a moderate level of parametric uncertainties for a given model. This will be demonstrated later in simulations by adding sensor noise and parametric uncertainties to the model. However, the form of the dynamics is assumed to be known exactly (as is usual with adaptive control and estimation algorithms).

The 6-Degree-of-Freedom equations of motion are well studied and can be found in [12], [2]. In this manuscript, we follow the notation and conventions from [2] by Beard and McLain. We shall not reproduce all the equations here and only refer to them where appropriate. For trigonometric ratios, we abbreviate them as $c_{(\cdot)}$ for $\cos(\cdot)$, and $s_{(\cdot)}$ for $\sin(\cdot)$.

The state of the system for this problem would consist of the quaternion-attitude of the MAV $x = [x_0\ x_1\ x_2\ x_3]^T$ (superscript $(\cdot)^T$ indicates transpose). We shall treat the angle-of-attack α

and the sideslip angle β as unknown parameters in the state dynamics that need to be estimated as part of the control solution. As indicated in the introductory section, available output measurements include x, the body-referenced angular velocity $y = [p\ q\ r]^T$, and the airspeed V_a. The control inputs are the aileron δ_a, elevator δ_e, and rudder δ_r, inputs for the attitude-control module presented in this letter, and a throttle δ_t for the outer loop to control the vehicle's airspeed (not covered in this letter). We also assume that the vehicle's aerodynamic model consisting of the stability and control derivatives, geometry, mass, moment-of-inertia, and the air density, is available.

The first step is to obtain the equations of motion for the vehicle's attitude. The dynamical equations are:

$$y = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = 2 \begin{bmatrix} -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & -x_1 & x_0 \end{bmatrix} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = 2A\dot{x},$$

$$\dot{x} = \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = 0.5 \begin{bmatrix} -x_1 & -x_2 & -x_3 \\ x_0 & -x_3 & x_2 \\ x_3 & x_0 & -x_1 \\ -x_2 & x_1 & x_0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{2} A^T y,$$

$$\ddot{x} = \begin{bmatrix} \ddot{x}_0 \\ \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \frac{1}{2} A^T \dot{y} + \frac{1}{2} \dot{A}^T y. \tag{1}$$

where the matrix $A(x)$ is a function of the state and satisfies $AA^T = 1_{3\times 3}$, $A^T A = 1_{4\times 4} - xx^T$. Now we use the equations of motion for the angular accelerations $\dot{p}$, $\dot{q}$, and $\dot{r}$. These contain a Coriolis term and the moment terms. The moments themselves are expressed in terms of the vehicle's aerodynamic coefficients. For a derivation of the following equation, the reader is referred to [2] chapter 4.

$$\dot{y} = \begin{bmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6(p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{bmatrix} + \frac{\rho V_a^2 S}{2} \begin{bmatrix} b\Gamma_3 & & b\Gamma_4 \\ & c\Gamma_9 & \\ b\Gamma_4 & & b\Gamma_8 \end{bmatrix}$$

$$\begin{bmatrix} C_{l0} + C_{l,\beta}\beta + (C_{l,p}p + C_{l,r}r)b/2V_a \\ cC_{m0} + C_{m,\alpha}\alpha + C_{m,q}qc/2V_a \\ C_{n0} + C_{n,\beta}\beta + (C_{n,p}p + C_{n,r}r)b/2V_a \end{bmatrix}$$

$$+ \frac{\rho V_a^2 S}{2} \begin{bmatrix} b\Gamma_3 & & b\Gamma_4 \\ & c\Gamma_9 & \\ b\Gamma_4 & & b\Gamma_8 \end{bmatrix} \begin{bmatrix} C_{l,\delta a}\delta_a + C_{l,\delta r}\delta_r \\ C_{m,\delta e}\delta_e \\ C_{n,\delta a}\delta_a + C_{n,\delta r}\delta_r \end{bmatrix}, \tag{2}$$

$$= D(x, \dot{x}) + GH(\alpha, \beta) + GIy(x, \dot{x}) + GJ\delta, \tag{3}$$

where, the Γ_i's are all related to the moments-of-inertia; $\delta = [\delta_a\ \delta_e\ \delta_r]^T$ is the attitude control input vector; b, c, and S are the aircraft's wingspan, chord length, and planform area; ρ is the air density; $C_{(\cdot)}$ are the stability and control derivatives; subscripts l, m, n denote the rotation x, y, and z-axes; and the matrices $A, D, G, H, \ldots$ are suitably defined below. We have obtained the first term on the RHS in (1). We next determine

the second term. To do this in an elegant manner, we shall use quaternion algebra (see [12] chapter 11 section 6). Let $\otimes$ denote quaternion multiplication. Note that $\dot{x} = A^T y/2$ is nothing but $(x \otimes y)/2$. So, $\ddot{x} = (x \otimes \dot{y} + \dot{x} \otimes y)/2 = (x \otimes \dot{y})/2 + (x \otimes y \otimes y)/4$. The first term is just $A^T \dot{y}/2$ in matrix notation, and we already have an expression for $\dot{y}$ in (3). For the second term, we note that $y \otimes y = -\|y\|^2$. Thus, we obtain:

$$\dot{A}^T y/2 = -(p^2 + q^2 + r^2)x/4 = -\|y(x, \dot{x})\|^2 x/4. \tag{4}$$

Substituting from (3) and (4) in (1), we obtain:

$$\ddot{x} = (\dot{A}^T y + A^T \dot{y})/2 = -\|y(x, \dot{x})\|^2 x/4 + A^T(x)(D(x, \dot{x})$$
$$+ GH(\alpha, \beta) + GIy(x, \dot{x}) + GJ\delta)/2, \tag{5}$$

where (blank entries imply zero values)

$$D = \begin{bmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6(p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{bmatrix}; \quad G = \frac{\rho V_a^2 S}{2} \begin{bmatrix} b\Gamma_3 & & b\Gamma_4 \\ & c\Gamma_9 & \\ b\Gamma_4 & & b\Gamma_8 \end{bmatrix};$$

$$H = \begin{bmatrix} C_{l0} + C_{l,\beta}\beta \\ C_{m0} + C_{m,\alpha}\alpha \\ C_{n0} + C_{n,\beta}\beta \end{bmatrix}; \quad I = \frac{1}{2V_a} \begin{bmatrix} bC_{l,p} & & bC_{l,r} \\ & cC_{m,q} & \\ bC_{n,p} & & bC_{n,r} \end{bmatrix};$$

$$J = \begin{bmatrix} C_{l,\delta a} & & C_{l,\delta r} \\ & C_{m,\delta e} & \\ C_{n,\delta a} & & C_{n,\delta r} \end{bmatrix}. \tag{6}$$

While having expanded the matrices for the sake of completeness, the essence of (5) is, however, the form of the quaternion attitude dynamics. It can be seen that (5) expresses $\ddot{x}$ in terms of the following additive terms:

- a "centrifugal" term $-\|y\|^2 x/4$
- a Coriolis term $A^T(x)D(x, \dot{x})/2$
- a disturbance term $A^T(x)GH(\alpha, \beta)/2$
- a dissipative term $A^T(x)GIy(x, \dot{x})/2$, and
- a control term $A^T(x)GJ\delta/2$

where, the matrices $G = G^T$ (related to the inverse of the moments-of-inertia matrix) and I (containing the damping derivatives) are known to be positive and negative definite respectively. In this form, the attitude dynamics are affected by the translational degrees of freedom only through the relative airspeed V_a, and the additive disturbance term $A^T GH$. The former, V_a, can be estimated from measurements, but the latter may contain unmodelled dynamics, and even otherwise, is difficult to estimate. In particular, it is the matrix H which contains α and β that is exogenous to the attitude dynamics as it is affected by an unknown external wind. The nonlinear dynamics and the presence of an unknown additive term multiplied by a state-dependent matrix motivate our choice for the method of solution, adaptive control and estimation, to our original problem of attitude control.

III. ADAPTATIVE ATTITUDE CONTROL AND DISTURBANCE ESTIMATION

Having realized the equations governing a fixed-wing MAV's attitude, we now turn to the problem of controlling the attitude

in the presence of significant wind. Equations (5) and (6) have reduced the equations of motion to differential equations expressing the angular accelerations ($\ddot{x}$) in terms of the angles (x), angular velocities ($\dot{x}$), several state- and geometry-dependent matrices (A, D, G, I, J), the measured airspeed V_a, and the unknown matrix H (or equivalently, the unknown aerodynamic angles α, β). If we make the quasistatic assumption that the response time of the attitude controller is much smaller than the characteristic times of the system and disturbance dynamics, then the matrices all remain quasistatic and we can use Lyapunov-based nonlinear adaptive estimation and control.

A brief clarification on the quasistatic assumption follows: this assumption does not imply that the dynamics of the external disturbances or the system are quasistatic relative to the other. In fact, the equations of motion strongly couple the system and disturbance dynamics. What the assumption does imply is that the controller must have a sufficiently fast reponse time. As we shall see in the results section, the controller response time is of the order of a tenth of a second. The assumption is then that the system and external disturbance (in the form of wind) do not vary significantly within about a tenth of a second. The natural vehicle dynamics of the vehicle are usually of the order of a second (for a vehicle of the size of the Aerosonde), and the power spectral density of a Dryden model gust at the relevant altitude and velocity scales is insignificant above 1 Hz [13]. Therefore, the quasistatic assumption is not prohibitive for the feasibility of the controller. With this remark, we return to the design of an adaptive attitude controller. We first show that the angular velocity y remains bounded.

Boundedness of the angular velocity: The boundedness of the angular velocity y and the time-derivative of the state $\dot{x}$ can be established under much more general conditions, without restriction to any specific controller design: *viz* as long as the actuation and disturbances are bounded and there is dissipation in the system, such assumptions being true for any real physical system. This result is easier to prove using vector notation. Let the moment-of-inertia matrix be denoted by N (equal to the inverse of G in (6) apart from the scaling factors $\rho V_a^2 S/2$, b, and c) and let $[y \times]$ denote the cross product matrix corresponding to vector y. Then,

$$N\dot{y} = [y \times]Ny + P + Qy,$$

where P is a bounded vector containing the disturbance and control terms (corresponding to $GH + GJ\delta$ in (3)), and Q is a negative definite damping matrix (corresponding to GIy in (3)). Taking the scalar product of y with this results in:

$$y^T N\dot{y} = y^T P + y^T Qy.$$

Using the well-known result from linear algebra that the quadratic forms involving positive definite matrices can be bounded on both sides by two positive numbers, we obtain the result that

$$\frac{\lambda_{N,\min}}{2} \frac{d\|y\|^2}{dt} \le y^T P - \lambda_{Q,\min}\|y\|^2,$$

where $\lambda_{P,\min}$ and $\lambda_{Q,\min}$ are the smallest eigenvalues of the positive definite matrices P and $-Q$. But this implies (using

[14] theorem 4.18) that y is bounded. Since $\dot{x} = A^T y/2$, with $\|A^T\|_2^2 = 3$, so is $\dot{x}$ bounded. We shall number this result for future reference:

$$y, \dot{x} \text{ are bounded}, \tag{7}$$

provided the actuation and disturbances are bounded, and there is positive dissipation (i.e., Q is negative definite).

Adaptive control problem formulation and design of the controller: For our design, we trim for a coordinated turn around a circle of radius R, at an airspeed of V_a, and rising (or dropping, for negative values) at a climb angle of γ. The coordinated turn covers a significant portion of a non-aerobatic vehicle's flight envelope. The three quantities R, V_a, and γ are translated to a desired attitude configuration $x_r(t = 0)$ using a trust-region algorithm (*cf* Matlab's fsolve function). This quaternion configuration is effectively the reference input to the attitude control module at time $t = 0$ (reference inputs are denoted by a subscript as $(\cdot)_r$). The reference attitude as a function of time can be obtained from the initial condition and the coordinated turn parameters:

$$x_r(t) = \begin{bmatrix} c_{\omega t/2} & 0 & 0 & -s_{\omega t/2} \\ 0 & c_{\omega t/2} & -s_{\omega t/2} & 0 \\ 0 & s_{\omega t/2} & c_{\omega t/2} & 0 \\ s_{\omega t/2} & 0 & 0 & c_{\omega t/2} \end{bmatrix} x_r(0),$$

where $\omega = V_a c_\gamma / R$ is the angular velocity of the coordinated turn. Let e_1 and e_2 denote the error in the attitude and its filtered dynamics respectively. Further let L_1 and L_2 be two symmetric positive definite gain matrices. In our analysis we shall assume they are just the identity multiplied by a positive scalar (which shall be denoted by the same symbols L_1 and L_2), so they commute with any other matrix.

$$e_1 = x - x_r, \ \dot{e}_1 = \dot{x} - \dot{x}_r = -L_1 e_1 + e_2.$$

$$\Rightarrow e_2 = \dot{x} - \dot{x}_r + L_1 e_1,$$

$$\text{and } \dot{e}_2 = -\|y\|^2 x/4 + A^T(D + GH + GIy + GJ\delta)/2$$

$$- L_1^2 e_1 + L_1 e_2 - \ddot{x}_r.$$

Let $\hat{H}$ be an estimate for H (which invokes α and β) and let $\tilde{H} = \hat{H} - H$. Assuming the control derivatives matrix J is of full row rank, i.e. we have a fully-actuated vehicle, and knowing that G is positive definite, we design the control input δ to cancel the known or measurable terms according to:

$$GJ\delta = -(D + G\hat{H} + GIy)$$

$$+ 2A(L_1^2 e_1 - (L_1 + L_2)e_2 + \ddot{x}_r) - 2\dot{A}e_2, \tag{8}$$

thus obtaining

$$2A\dot{e}_2 + 2\dot{A}e_2 = -2L_2 Ae_2 - G\tilde{H}, \tag{9}$$

where we have left-multiplied by $2A$ and used the fact that $Ax = 0$ and $AA^T = 1_{3 \times 3}$. The rank-three control authority enables the tracking of three rotational degrees of freedom. This should however suffice to control the four component attitude quaternion, which has one scalar degree of redundancy.

Looking at (9), the appropriate choice for the three basis degrees of freedom of error that can be regulated is given by Ae_2.

Equation (9) suggests the adaptation law, for some positive-definite matrix M, as

$$\dot{H} = \dot{\tilde{H}} = MG^T Ae_2/2, \tag{10}$$

and the below candidate Lyapunov function (CLF):

$$V_L = \frac{1}{2}\left(e_2^T A^T Ae_2 + \tilde{H}^T M^{-1}\tilde{H}\right). \tag{11}$$

So, $\dot{V}_L = e_2^T A^T\left(-L_2 Ae_2 - G\tilde{H}/2\right) + \tilde{H}^T\left(G^T Ae_2/2\right)$

$$= -L_2 e_2^T A^T Ae_2 = -L_2 e_2^T(1 - xx^T)e_2, \tag{12}$$

where we have used the quasistatic assumption that $\dot{H} = 0$ in (10), and the result that $A^T A = 1 - xx^T$. The CLF is therefore a positive definite function of e_2 and $\tilde{H}$, while its time derivative is a negative semi-definite function of e_2. From Lyapunov's stability theory ([14] chapter 8 lemma 8.2, theorem 8.4), this implies that V_L is a monotonically decreasing function that approaches a limit, say $V_{L\infty}$, and that e_2 and $\tilde{H}$ remain bounded for all time. The boundedness of e_2 implies the boundedness of $\dot{x}$ and y, assuming that the reference trajectory $\dot{x}_r + L_1 x_r$ is bounded. From (8), δ is then bounded, and from (2), (5), $\dot{y}$, $\ddot{x}$ are also bounded. For later reference, these results are numbered below:

$$\tilde{H}, e_2, \dot{x}, y \text{ are bounded, } \lim_{t\to\infty} V_L = V_{L\infty},$$

$$\delta, \dot{y}, \ddot{x}, \dot{e}_2 \text{ are bounded.} \tag{13}$$

Asymptotic convergence of orthogonal complement of filtered tracking error e_2: We now compute $\ddot{V}_L$

$$\ddot{V}_L = -2L_2 e_2^T A^T(\dot{A}e_2 + A\dot{e}_2)$$

$$= L_2 e_2^T A^T(2L_2 A_2 e_2 + G\tilde{H}). \tag{14}$$

Boundedness of e_2, $\tilde{H}$ (result (13)), A, and G yields bounded $\ddot{V}_L$, and by Barbalat's lemma ([14] chapter 8 lemma 8.2) we obtain

$$\lim_{t\to\infty}\dot{V}_L = \lim_{t\to\infty}(1 - xx^T)e_2 = 0,$$

$$\Rightarrow e_2 = k(t)x + \epsilon, \tag{15}$$

where $k(t) = x^T(t)e_2(t)$ is some scalar function of time, $\epsilon = (1 - xx^T)e_2$ is the orthogonal complement of e_2 with respect to x, and $\lim_{t\to\infty}\epsilon = 0$. In other words, e_2 gradually aligns with x, as it equals its projection $xx^T e_2$ on x. At this stage, we only know that the orthogonal complement of e_2 with respect to x goes to zero. However, x and x_r are not any arbitrary 4-vectors. Specifically, they are unit quaternions, and they follow the prescribed dynamics. These let us actually conclude that $\lim_{t\to\infty}e_2 = 0$ as shown next.

Asymptotic convergence of filtered tracking error e_2: Now, let us express e_2 in (15) as $\dot{x} - \dot{x}_r + L_1(x - x_r)$. Then we obtain a linear first order differential equation for x:

$$\dot{x} + (L_1 - k)x = \dot{x}_r + L_1 x_r + \epsilon. \tag{16}$$

Evaluating the evolution of $x^T x_r$ with time yields

$$d(x^T x_r)/dt = \dot{x}^T x_r + x^T \dot{x}_r$$

$$= (\dot{x}_r + L_1 x_r + \epsilon - (L_1 - k)x)^T x_r$$

$$+ x^T(\dot{x} + (L_1 - k)x - L_1 x_r - \epsilon)$$

$$d(1 - x^T x_r)/dt = -(2L_1 - k)(1 - x^T x_r) - x_r^T\epsilon, \tag{17}$$

If $2L_1 - k$ remains above a positive scalar μ, we can now apply the comparison lemma ([14] lemma 3.4) with respect to the linear constant coefficient differential equation $du/dt = -\mu u - x_r^T\epsilon$ which represents a stable filter applied on an asymptotically zero input, and conclude that the nonnegative quantity $1 - x^T x_r \to 0$. The only way the inner product of two unit quaternions can approach 1 is if they equal each other. Thus we are led to the conclusion that $x \to x_r$ as $t \to \infty$ provided $2L_1 - k$ remains above a positive scalar. The latter is quite easy to achieve, if, for example, the initial attitude $x(0)$ is chosen to lie within the hemisphere containing $x_r(0)$, and we choose $L_1 > \|\dot{x}_r(0)\| + \|\tilde{H}(0)\|/\lambda_{\min, M}$. Further, if $x \to x_r$, and $\ddot{x} - \ddot{x}_r$ is bounded, then we must also have $\dot{x} \to \dot{x}_r$, and so $e_2 \to 0$. This proves the asymptotic convergence of the attitude to the desired trajectory.

$$x \to x_r, \quad e_2 \to 0 \tag{18}$$

Asymptotic convergence of disturbance estimation: In the above derivations, we could only prove that $\tilde{H}$ is bounded. However, more can be said about $\tilde{H}$ on the basis of the dynamical equations under the quasistatic assumption. The dynamics of e_2 under the control law (8) are given by:

$$\dot{e}_2 = -\|y\|^2 x/4 - A^T G\tilde{H}/2 - A^T \dot{A}e_2 - A^T AL_2 e_2$$

$$+ xx^T(-L_1^2 e_1 + L_1 e_2 - \ddot{x}_r).$$

The derivatives of all quantities appearing on the right-hand-side are bounded, which shows that $\ddot{e}_2$ also remains bounded. On the other hand, we already know that $e_2 \to 0$. Hence by Barbalat's lemma, $\dot{e}_2 \to 0$, and so from (9):

$$\tilde{H} = -2G^{-1}(A\dot{e}_2 + L_2 Ae_2 + \dot{A}e_2) \to 0, \tag{19}$$

and we are led to the conclusion that the asymptotic convergence of e_2 to zero also implies the asymptotic convergence of $\tilde{H}$ to zero.

A more efficient controller: The efficiency of the controller presented in (8) can be improved by accounting for the fact that the dissipation matrix I is negative in most physical systems and instead using the control law:

$$GJ\delta = -(D + G\hat{H} + GIA(\dot{x}_r + L_1 x_r))$$

$$+ 2A(L_1^2 e_1 - (L_1 + L_2)e_2 + \ddot{x}_r) - 2\dot{A}e_2, \tag{20}$$

which differs from (8) in the term $-GIy = -2GIA\dot{x}$ being replaced by the term $-GIA(\dot{x}_r + L_1 x_r)$. If we perform Lyapunov stability analysis on the CLF $V_{L2} = (e_2^T A^T G^{-1} Ae_2 + \tilde{H}^T M^{-1}\tilde{H})/2$, we obtain $\dot{V}_L = -e_2^T A^T(L_2 G^{-1} - I)Ae_2$, where I being negative definite leads to faster convergence of e_2 to zero while requiring lesser controller effort.

The assumptions that are needed to obtain the results presented in this section are listed below:

1) The moment of inertia matrix N, its inverse G are positive definite.
2) The vehicle is fully actuated, i.e., matrix J has full row rank so that we can generate three independent moments along the three axes.
3) The system has dissipative moments (friction) that prevent build-up of excessive kinetic energy. In a linear model of the dissipation, matrix I in (5) must be negative definite.
4) There are bounds on the physical actuators, i.e., δ is bounded.
5) The external disturbances are bounded, i.e., H is bounded.
6) The desired trajectory, x_r, and its time derivative, $\dot{x}_r$ are bounded.
7) The disturbances are quasistatic relative to the controller's response time: $\dot{H} \approx 0$.
8) The geometric, inertial, and aerodynamic coefficients in matrices G, I and J are known.

Other than the last two, all the remaining assumptions are valid in almost all engineering problems. The quasistatic assumption 7 has already been discussed at length at the beginning of Section III. Referring to assumption 8, it is straight forward to obtain the geometric and inertial coefficients to a good accuracy. The aerodynamic coefficients could be obtained using a CFD program such as the Athena Vortex Lattice (AVL) program [15]. The coefficients returned by such a program are prone to deviate from the true values. We therefore need to verify that the controller is robust to errors in the aerodynamic coefficient matrices.

A final remark on the estimation of α and β follows: A second approach to estimate the aerodynamic angles is to measure the translational accelerations and derive the α-, β-dynamics from them. The relevant definitions would be $\alpha = \text{atan}(w_r/u_r)$, and $\beta = \text{asin}(v_r/V_a)$, where $[u_r\ v_r\ w_r]^T = [u - u_w\ v - v_w\ w - w_w]^T$ is the relative velocity of the vehicle, subscripts $(\cdot)_r$ and $(\cdot)_w$ denoting relative and wind quantities respectively, and $V_a^2 = u_r^2 + v_r^2 + w_r^2$. From this we can obtain:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} (u_r \dot{w}_r - w_r \dot{u}_r)/(u_r^2 + w_r^2) \\ (V_a \dot{v}_r - v_r \dot{V}_a)/(V_a \sqrt{V_a^2 - v_r^2}) \end{bmatrix}, \qquad (21)$$

Looking at the dynamics, we see that the quantities available at our disposal are the measured relative airspeed V_a, estimates of α and β, the measured angular velocity components $[p\ q\ r]^T$ and the measured translational acceleration $[a_x\ a_y\ a_z]^T$. Expressing (21) in terms of these quantities, we obtain:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} q + c_\alpha \left(-ps_\beta + a_z/V_a\right)/c_\beta - s_\alpha \left(rs_\beta + a_x/V_a\right)/c_\beta \\ ps_\alpha - rc_\alpha - s_\beta c_\alpha a_x/V_a + c_\beta a_y/V_a - s_\beta s_\alpha a_z/V_a \end{bmatrix}$$
$$(22)$$

where we have ignored the unknown wind quantities. These process dynamics of α and β yield another set of estimates, which could be fused with the estimates obtained by using (10) if the form of disturbance dynamics in matrix H is exactly known in terms of α and β. In this letter, we use only the adaptively estimated effects of α and β in order to keep the
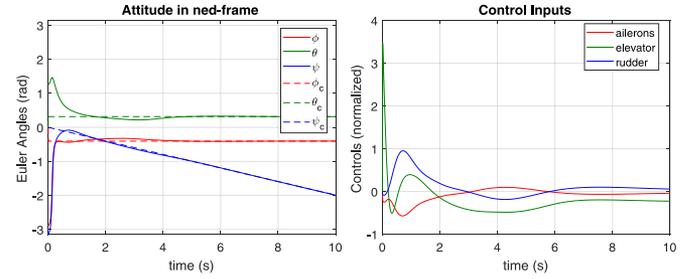

Fig. 2. Attitude tracking from a large initial error and in the absence of disturbances. The left subfigure shows tracking performance and the right subfigure shows the associated adaptive control.

development concise and to remain open to the possibility that the disturbance dynamics may not be entirely known.

So far, we have imposed realistic bounds upon actuation, but not checked if the control law in equation (8) respects these bounds. Since the control law depends not only upon the state and its time derivative, but also on the effect of the disturbances, these considerations are best verified in simulations to check the tolerable magnitude of the wind disturbance that keeps the controller within bounds. This is what we proceed to do in the next section.

IV. SIMULATION RESULTS

In this section, we evaluate the performance of the controller designed in (8) and (10) in Matlab with respect to tracking error, control magnitude, and estimation errors. We use the Aerosonde model as a representative MAV [2]. For simplicity, we shall assume that elements of the disturbance matrix H are specifically of the form given in (6), being affine functions of the aerodynamic angles. The objective is to control the attitude to track a reference trajectory and to estimate the disturbances, or equivalently, the aerodynamic angles. In the figures, the attitude is displayed in terms of Euler angles for better readability. The controls are normalized with respect to their bounds, so that they saturate at ± 1.

Since the coordinated turn covers most of a non-aerobatic aircraft's flight envelope, it is chosen as the prescription for the reference trajectory. The coordinated turn is characterized by its radius $R = 100$ m, the vehicle's relative airspeed $V_a = 20$ ms^{-1}, and a climb angle of $\gamma = 5°$. The vehicle is initialized at an attitude that has a random error of standard deviation 1rad from the trim attitude. The attitude of the vehicle is plotted along with the reference trajectory in Fig. 2(a), followed by the control in 2(b). It can be seen that the attitude stabilizes from an initial error as large as π rad. The controller yields asymptotic tracking even with actuator saturation enabled as seen in Fig. 3.

We now introduce a disturbance in the form of a sinusoidal wind. Since the dynamical equations are nonlinear, we cannot use superposition to add the vehicle's response to disturbances at multiple frequencies. The choice of a sinusoidal disturbance has more to do with using a smooth, bounded disturbance that does not vanish to zero with time. The frequency of the wind is chosen so that it excites the maximum response from the vehicle for small disturbances. This is usually a function of the inertial and
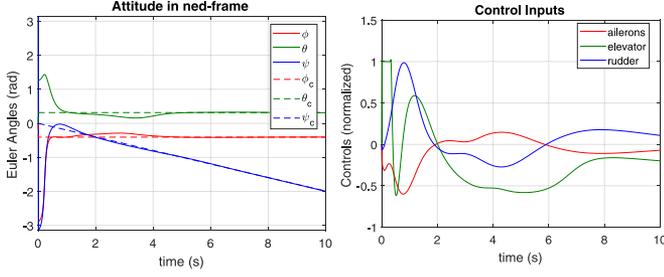
Fig. 3. Attitude tracking from a large initial error and in the absence of disturbances with actuator saturation enforced. The left subfigure shows tracking performance and the right shows the associated adaptive control.
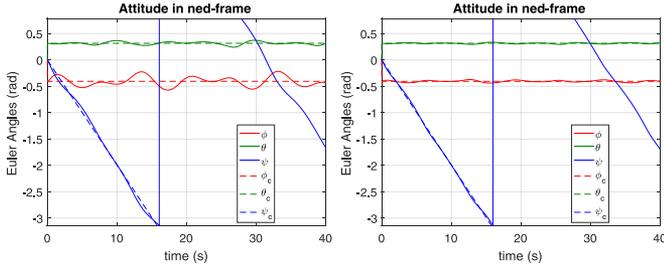

Fig. 4. Tracking errors without (left subfigure) and with (right subfigure) adaptive estimation of α and β. Notice the tighter control over the attitude with adaptation.
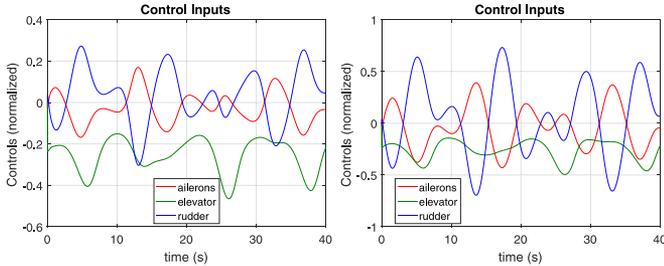

Fig. 5. Control input without (left subfigure) and with (right subfigure) adaptive estimation of α and β. Notice the richer control against the wind disturbance when estimation is enabled, which yields the better tracking performance in Fig. 4.
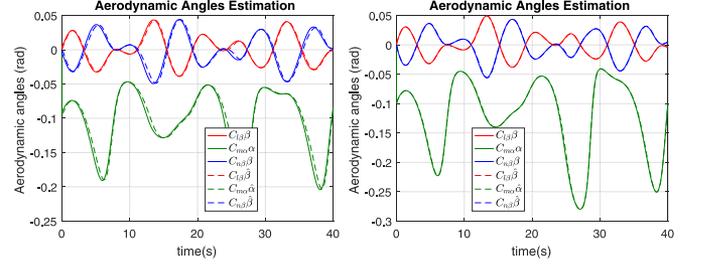

Fig. 6. Disturbance estimation with nominal (left subfigure) and tuned (right subfigure) adaptation gains. Tuning the gains leads to a controller response time lesser than 0.1 s.
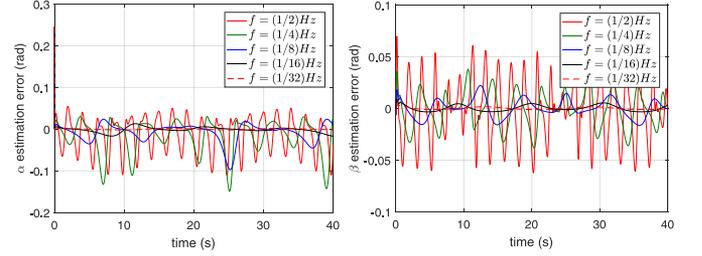

Fig. 7. Estimation error in α (left subfigure) and β (right subfigure) for disturbance frequency ranging from $(1/2)$Hz to $(1/32)$Hz..
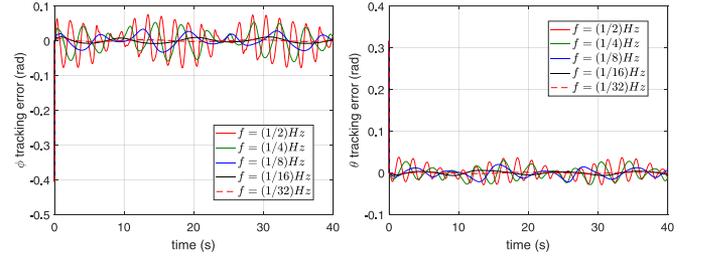

Fig. 8. Tracking error in roll (left subfigure) and pitch (right subfigure) attitude for wind frequency ranging from $(1/2)$Hz to $(1/32)$Hz.

aerodynamic coefficients of the vehicle. By performing a small signal sinesweep analysis about trimmed flight, it was seen that the vehicle exhibited peak aerodynamic response to stimulus close to $(1/8)$ Hz. Accordingly, The vehicle is disturbed by a westward wind (along positive y-axis) of $6\sin(2\pi t/8)\,\mathrm{ms}^{-1}$.

Fig. 4 shows the errors in the vehicle's attitude without and with the adaptive estimation of the disturbance matrix $H(\alpha, \beta)$. The estimation is effectively disabled by setting the adaptation gains, matrix M in (10), to zero. We can see that the roll and yaw attitude are regulated to the trim values much better when estimation is enabled. The next figure, Fig. 5, shows a stronger aileron and rudder control against the wind disturbance using the estimated $\hat{H}$ but still within the bounds of ± 1. Without the estimation, the aileron and rudder controls remain subdued. The estimate $\hat{H}$ (Fig. 6) converges to its true values $H(\alpha, \beta)$ within a tenth of a second with the right adaptation gains. This justifies

the quasi-static assumption for all disturbances of time scales greater than a second.

Having evaluated the typical performance of the controller, we now proceed to evaluate its sensitivity to sensor noise, parameter uncertainty, and disturbance frequency. We first check the estimation and tracking errors across frequencies for a fixed amplitude of the disturbance.

It can be seen in Figs. 7 and 8 that the errors progressively worsen at higher frequencies of excitation. However, we need to remember that the power spectral density of disturbances reduces with frequency. The Dryden model of turbulence, for example, predicts a $(1 + 3(\omega/\omega_0)^2)/(1 + \omega^2/\omega_0^2)^2$ frequency dependence of atmospheric turbulence in the lateral and vertical directions and a $1/(1 + \omega^2/\omega_0^2)$ frequency dependence of atmospheric turbulence in the longitudinal direction [13]. Here, ω_0 is the natural frequency associated with the flight, and depends upon the altitude of flight and the relative airspeed.

Next, we include sensor noise (of 0.1 rad/$\sqrt{\mathrm{Hz}}$ in the attitude sensor) and parametric uncertainties in the model (of about 20% errors in the control and stability derivatives). This shows an
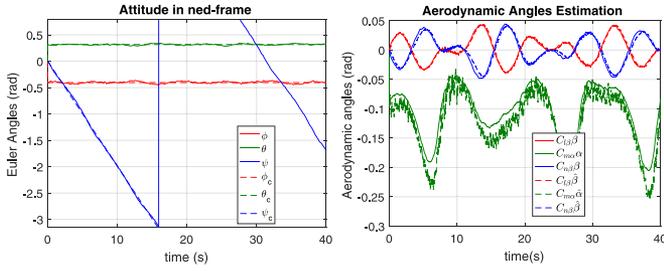
Fig. 9.   Effect of $0.1 \mathrm{rad/s/}\sqrt{\mathrm{Hz}}$ sensor noise and $20\backslash\%$ parametric uncertainties on attitude tracking (left subfigure) and estimation (right subfigure).
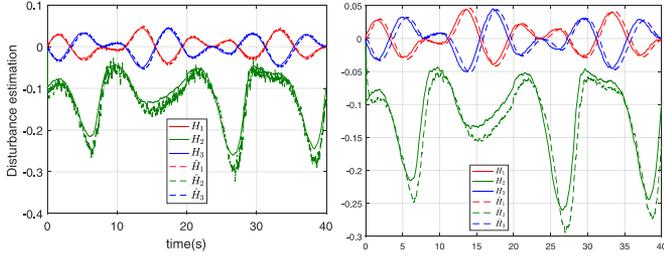

Fig. 10.   Noise in real sensors requires the presence of a filter that rejects the noise while preserving the signal. The unfiltered estimation (left figure) is smoothened out using a second order IIR filter (right subfigure).
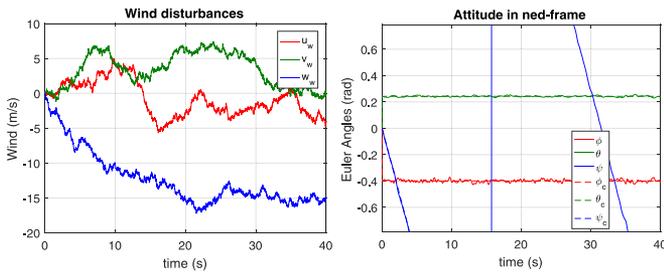

Fig. 11.   Controller performance in the presence of a stochastic wind prescribed by the Dryden wind turbulence model. The subfigure on the left shows the prescribed wind disturbance, and the subfigure on the right shows the tracking.

expected degradation in tracking and estimation performance (Fig. 9). The chatter in the estimator in the presence of sensor noise can be filtered out. Fig. 10 shows the improvement in the estimation upon including a second order IIR (Infinite impulse response) filter that rejects noise above 10Hz.

The attitude dynamics derived in Section II were nonlinear in terms of the state and its derivative. Therefore, we cannot use superposition to infer the performance of the controller from the results to individual frequency components. Therefore, the controller is also evaluated against a stochastic disturbance that is prescribed using Dryden's wind turbulence model [13]. The results for a typical simulation are shown in Fig. 11.

V. CONCLUSION

We have thus summarized a different approach to control MAVs, where we first control the attitude using a nonlinear controller, and then use an outer feedback loop to control the translational velocities in order to track any desired trajectory. This may be contrasted with the conventional lateral and longitudinal split and linear controller designs. We justify this approach to solve the problem of controlling MAVs flying at small relative speeds. The main challenge that arises in this approach is the coupling into the attitude dynamics from the linear velocities in the form of the aerodynamic angles α and β. We solve this problem by using Lyapunov-based methods to adaptively estimate the effects of α and β. By using arguments from Lyapunov's stability theory we show that it is possible to achieve asymptotic attitude tracking and disturbance estimation under mild assumptions. The controller development is finally verified in simulations that validate the theoretical results of robustness and adaptability.

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